Vol. 51, No. 3 DUKE MATHEMATICAL JOURNAL © September 1984

# SINGULAR GREEN OPERATORS AND THEIR SPECTRAL ASYMPTOTICS

GERD GRUBB

## 1. Introduction.

1.1. In the paper [7], L. Boutet de Monvel introduced a complete calculus of boundary value problems associated with pseudo-differential operators $P$ on an $n$-dimensional $C^\infty$ manifold $\Sigma$, having the so-called transmission property at the boundary $\partial\Omega$ of a smooth open subset $\Omega$. The calculus builds on earlier works by Vishik and Eskin, but introduces in a systematic way an interesting new tool, the singular Green operators, that appear naturally also in considerations of differential operators. The theory has been applied by rather few people (let us mention the index theory in [7] and in works of Rempel and Schulze [27], [28], ..., and the spectral theory in [11], [13], [15]), which perhaps has something to do with the fact that the original presentation in [7] is very sketchy, leaving a burden of proof to the user. For instance, composition formulas are proved in [7] only in the simplest constant coefficient cases. (The book [28] gives many more details, but passes lightly over the composition of fully $x_n$-dependent symbols.)

During our studies of parameter-dependent symbols [16], [17], we found it necessary to develop our own treatment of parts of the (nonparametrized) theory, and in particular to improve the composition formulas. The present paper first describes some central parts of the calculus, where we take a more "real" point of view than [7] and [28], working with kernels as well as symbols, since this makes the theory more accessible. Then we establish (in the variable coefficient case) a new and simple formula for the most complicated composition rule that is concerned with a certain "leftover" operator in the composition of two truncated pseudo-differential operators. This is a singular Green operator (s.g.o.), acting on $\overline{\Omega}$ but not of a ps.d.o. type (and not in itself "elliptic").

Next, we study the spectral properties of general s.g.o.s $G$ on a compact manifold $\overline{\Omega}$, and show that when $G$ is of order $-d<0$ (and class 0), then its characteristic values $s_k(G)=\lambda_k(G^*G)^{1/2}$ satisfy the asymptotic estimate

$$s_k(G)k^{d/(n-1)} \to C(g^0) \qquad \text{for} \quad k\to\infty, \tag{1.1}$$

where $C(g^0)$ is a constant derived from the principal symbol $g^0$; this uses the precise calculus. Finally, to illustrate further the usefulness for differential

Received September 9, 1982. Revision received March 1, 1984.