Vol. 51, No. 2 DUKE MATHEMATICAL JOURNAL © June 1984

A COUNTEREXAMPLE IN $H^\infty + BUC$

CARL SUNDBERG

§1. Introduction. Let H^∞ denote the space of bounded analytic functions in the unit disc $D = [|z| < 1]$. By Fatou's Theorem H^∞ can be identified via radial limits with a closed subalgebra of $L^\infty(\partial D)$. The closed subalgebras between H^∞ and L^∞ are called *Douglas algebras* and have been studied e.g., in [1], [2], [3], [8], [14], [16].

It was shown by Sarason [10] that $H^\infty + C$, the linear span of H^∞ and the continuous functions on ∂D, is a Douglas algebra. In [12] he raised the question of whether a unimodular function in $H^\infty + C$ could always be written as the product of an inner function and a function in $QC \underset{\text{def}}{=} (H^\infty + C) \cap (\overline{H^\infty + C})$. This question was answered affirmatively by Wolff in [17]. It was pointed out by Hayashi [6] that this question is relevant to a problem in prediction theory considered earlier by Helson and Sarason. We will now very briefly discuss this problem.

Let $w \in L^1(\partial D)$, $w \geqslant 0$, let $\mathscr{P}_0$ denote the closed linear span in $L^2(w\,d\theta)$ of $\{1, e^{-i\theta}, e^{-2i\theta}, \dots\}$, and let $\mathscr{F}_n$ denote the closed linear span in $L^2(w\,d\theta)$ of $\{e^{in\theta}, e^{i(n+1)\theta}, \dots\}$. We are interested in conditions on w which assure that the angle between $\mathscr{P}_0$ and $\mathscr{F}_n$ goes to $\pi/2$ as $n \to \infty$ (so that w is the spectral density of a *strongly mixing* discrete stationary Gaussian process, see [4]). As shown by Helson and Sarason [7], a necessary and sufficient condition for this to happen is that $w = |H|$ where H is an outer function in $H^1(\partial D)$ and $\overline{H}/|H| \in H^\infty + C$. They then show that this condition is equivalent to the statement that w can be written as $w = |P|^2 e^{u+\tilde{v}}$ where P is a trigonometric polynomial, u and v are continuous, and $\tilde{v}$ is the Hilbert transform of v (see also [11]). It is implicit in Hayashi's paper that this equivalence follows easily from Wolff's factorization theorem.

Consideration of continuous Gaussian processes leads one to ask the analogous question on the real line R. Hayashi in the above paper studies this and shows that the strongly mixing condition is equivalent to $w = |H|$ where $H \in H^1(\mathsf{R})$ is outer and $\overline{H}/|H| \in H^\infty + BUC$, the linear span of (boundary values of) bounded analytic functions on $\Delta = \{z = x + iy : y > 0\}$ and the space BUC of bounded uniformly continuous functions on R. The space $H^\infty + BUC$ was shown to be a Douglas algebra by Sarason [13]. The natural generalization to this situation of the representation of w obtained by Helson and Sarason would be $w = |P|^2 e^{u+\tilde{v}}$, where now P is an entire function of exponential type in $L^2(\mathsf{R})$, and u and v are in BUC. Hayashi showed that if $w = |H|$ as above with

Received August 12, 1983.