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A BOUND ON THE DIMENSIONS OF COMPLETE SUBVARIETIES OF $\mathcal{M}_g$

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Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus $g \geqslant 3$. In this article we prove that if $Z \subset \mathcal{M}_g$ is a complete subvariety, then dimension $Z \leqslant g-2$. This of course is immediately equivalent to the corollary that if Y is complete and $\pi : X \to Y$ is a family of smooth curves of genus g such that for all points y in some open dense subset of Y $\pi^{-1}(y)$ is isomorphic to at most finitely many other fibers of π, then dimension $Y \leqslant g-2$. We also obtain bounds on the dimensions of complete subvarieties contained in certain other subvarieties of $\mathcal{M}_g$. It is not known whether these bounds are sharp. See [H] for a discussion of this question as well as some possible applications of the results of this paper.

The proof uses a stratification of $\mathcal{M}_g$ which was inspired by the stratification studied by Arbarello in [A1] and [A2]. Much of the idea of how to use such a stratification is also due to Arbarello. There still remains the interesting question of whether Arbarello's stratification can also provide a proof.

We work over the complex numbers.

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In a family of smooth curves of genus g, $\pi : X \to Y$ we assume that π is smooth, proper, and surjective. The family is complete if Y is complete, nondegenerate if the induced map $Y \to \mathcal{M}_g$ is finite, and generically nondegenerate if there is a dense open subset $U \subset Y$ such that the restricted map $U \to \mathcal{M}_g$ is finite.

It is already known that the only complete subvarieties of $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ are points so for the rest of the paper we assume $g \geqslant 3$.

Definition 1. $H_g(i,j) = \{[C] \in \mathcal{M}_g :$ there exists a finite map $f : C \to \mathbf{P}^1$ with degree $f \leqslant i$, and the number of points in $f^{-1}\{0,\infty\}$ not counting multiplicity is less than or equal to $j\}$.

Clearly there is a close relationship between $H_g(i,j)$ and Hurwitz schemes. In particular we will need to know some facts about the relationship between $H_g(i,j)$ and $\overline{H}_{k,B}$ a compactification of the Hurwitz scheme developed by Harris and Mumford [HM] and generalized by Diaz [D]. The reader not familiar with this object should read those articles before proceeding. We use the same notation as in [D]. The symbol $\overline{H}_{k,B}$ will be used to refer to both the Hurwitz scheme itself and its image in moduli space, Δ refers to points corresponding to

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