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# DIFFERENTIAL OPERATORS AND THE SINGULAR VALUES OF EISENSTEIN SERIES

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There are three main themes in this paper:

I. The values of holomorphic and nonholomorphic Eisenstein series of symplectic and unitary groups at $CM$-points;

II. The critical values of certain zeta functions;

III. Differential operators of arithmetic type on symmetric domains and their adjoint operators.

To explain our problems, let us first define the group, the domain, and the series in less general forms than what is actually treated in the text:

$$G = \left\{ \alpha \in SL_{2m}(K) \,|\, {}^t\bar{\alpha}\iota_m\alpha = \iota_m \right\}, \qquad \iota_m = \begin{bmatrix} 0 & -1_m \\ 1_m & 0 \end{bmatrix},$$

$$H = \left\{ z \in M_m(\mathbf{C}) \,|\, i({}^t\bar{z} - z) \text{ is positive definite} \right\},$$

where $K$ is an imaginary quadratic field. For $z \in H$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ with $a, b, c, d$ of size $m$, we put

$$\alpha(z) = (az + b)(cz + d)^{-1}, \qquad \eta(z) = i({}^t\bar{z} - z), \qquad \delta(z) = \det(\eta(z)),$$

$$\kappa_\alpha(z) = \bar{c} \cdot {}^t z + \bar{d}, \qquad \mu_\alpha(z) = cz + d, \qquad j_\alpha(z) = \det(cz + d),$$

and define two types of Eisenstein series as follows:

$$\mathbf{E}(z, s; k, r) = \delta(z)^{s-r} \sum_{\alpha \in (\Gamma \cap P)\backslash\Gamma} j_\alpha(z)^{-k-r}\overline{j_\alpha(z)}^{\,r}|j_\alpha(z)|^{-2s},$$

$$\mathbf{E}(z, s; k, \sigma) = \delta(z)^{s} \sum_{\alpha \in (\Gamma \cap P)\backslash\Gamma} j_\alpha(z)^{-k}|j_\alpha(z)|^{-2s}\sigma\left( \mu_\alpha(z)^{-1}\overline{\kappa_\alpha(z)}\,\eta(z)^{-1} \right).$$

Here $s \in \mathbf{C}$, $k \in \mathbf{Z}$, $0 \leqq r \in \mathbf{Z}$, $\Gamma$ is a congruence subgroup of $G$, $P$ is a parabolic subgroup of $G$ consisting of all $\alpha$ for which $c = 0$, and $\sigma$ is a $\overline{\mathbf{Q}}$-rational polynomial representation $GL_m(\mathbf{C}) \to GL_n(\mathbf{C})$. The former series can be obtained as a special case of the latter by taking $\sigma(x) = \det(x)^r$. For simplicity, let us write $\mathbf{E}(z, s)$ for any of these series.

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