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UNIFORM DISTRIBUTION OF HOROCYCLE ORBITS FOR FUCHSIAN GROUPS

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Let $G = \mathrm{SL}(2, \mathbf{R})$ and let Γ be a lattice in G; that is, a discrete subgroup such that G/Γ admits a finite G-invariant measure. By scaling we may assume that the induced measure on G/Γ which we shall denote by μ, has total volume 1. Groups Γ of this form are known classically as "Fuchsian groups of the first kind". In this paper we study the horocycle flows (i.e., action of the 1-parameter group $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $t \in \mathbf{R}$) on the space G/Γ.

When G/Γ is compact it was proved by Hedlund that the horocycle flow is minimal. This result was strengthened by Furstenberg who proved that the flow was strictly ergodic; in other words every orbit is uniformly distributed with respect to μ. (We note that simpler proofs have now been obtained for this result by various authors including R. Ellis and W. Perrizo [4] and more recently by D. S. Ornstein (oral communication).)

When G/Γ is not compact, the flow is not minimal. There are periodic orbits for the flow. Hedlund [6] proved that the flow is topologically transitive. In fact he showed that every horocycle orbit is either periodic or dense. In this paper we strengthen Hedlund's result by proving the following theorem.

THEOREM 1. *Let $G = \mathrm{SL}(2, \mathbf{R})$ and Γ be a lattice in G such that G/Γ is not compact. Then the nonperiodic orbits of the horocycle flow $\{h_t \mid t \in R\}$ are uniformly distributed with respect to μ; that is, if x_0 is in such an orbit then for any bounded continuous function f on G/Γ, as $T \to \infty$*

$$\frac{1}{T}\int_0^T f(h_t x_0)\,dt \to \int_{G/\Gamma} f\,d\mu$$

This theorem generalizes an earlier result of Dani [3] for $\Gamma = \mathrm{SL}(2, \mathbf{Z})$.

These horocycle flows exhibit what seems to be unusual behaviour for a dynamical system. An orbit is said to be *generic* if it is uniformly distributed with respect to some invariant measure. A periodic orbit is clearly generic; according to the theorem the remaining orbits are also generic. Thus the horocycle flow on G/Γ is an example of a topologically transitive but not minimal flow for which all orbits are generic.

Questions of uniform distribution may be raised for action of (iterates of) a single translation of G/Γ by say $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. Arguing as in [3] it is easy to deduce from the continuous version the following.

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