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PATH INTEGRAL FOR A HYPERBOLIC SYSTEM OF THE FIRST ORDER

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Introduction. The present paper concerns a path integral approach to the $N \times N$ hyperbolic system of the first order

$$\frac{\partial}{\partial t}\psi(t,x) = \left[\sum_{l=1}^{d} P_l\left(\frac{\partial}{\partial x_l} - iA_l(t,x)\right) + iQ + iV(t,x)\right]\psi(t,x)$$

$$0 < t < T, \quad x \in \mathbf{R}^d, \quad (0.1)$$

where $0 < T \leqslant \infty$, and the P_l, $1 \leqslant l \leqslant d$, and Q are constant $N \times N$-matrices, and the $A_l(t,x)$, $1 \leqslant l \leqslant d$, and $V(t,x)$ are real-valued functions. It is assumed not only that the P_l have only real eigenvalues but also that they are simultaneously diagonalizable. This assumption is satisfied for instance by the Dirac equation in two space-time dimensions.

The physical idea of path integral was introduced by Feynman ([11], [12]) to make an intuitive representation of the solution of the Schrödinger equation. With the Wiener measure on the Brownian path space Kac [15] has given a rigorous realization of Feynman's idea for the pure-imaginary-time Schrödinger equation, i.e., the heat equation to establish what is now called the Feynman–Kac formula.

The aim of the present paper is to construct, for each fixed $(t,x) \in [0,t) \times \mathbf{R}^d$, an $N \times N$-matrix-valued countably additive path space measure $\nu^0_{t,x}$ on the Banach space $C([0,t]; \mathbf{R}^d)$ of the continuous paths $X : [0,t] \to \mathbf{R}^d$ and to establish for the solution $\psi(t,x)$ of the Cauchy problem for the hyperbolic system (0.1) with initial datum $\psi(0,x) = g(x)$ the path integral formula

$$\psi(t,x) = \int d\nu^0_{t,x}(X)\exp\left[i\left(\int_0^t A(s,X(s))\,dX(s) + \int_0^t V(s,X(s))\,ds\right)\right]g(X(0)). \quad (0.2)$$

The support of $\nu^0_{t,x}$ is on the space of the Lipschitz continuous paths $X : [0,t] \to \mathbf{R}^d$ satisfying $X(t) = x$ and that

$$\text{for each } a, b \text{ with } 0 \leqslant a < b \leqslant t \text{ and each unit vector } \omega \text{ in } \mathbf{R}^d,$$
$$c_-(\omega)(b-a) \leqslant (X(b) - X(a))\omega \leqslant c_+(\omega)(b-a). \quad (0.3)$$

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