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COMMENSURABILITY OF CO-COMPACT THREE-DIMENSIONAL HYPERBOLIC GROUPS

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1. Introduction. Two groups Γ_1, Γ_2 are called *commensurable* if they have subgroups Δ_1, Δ_2 respectively, both of finite index, with $\Delta_1 \cong \Delta_2$. In the case of co-compact discrete groups of isometries of hyperbolic 3-space H^3, it would then follow from Mostow's Theorem [5] that Δ_1 and Δ_2 are conjugate in PSL(2, C). In this paper, we shall work with subgroups of SL(2, C) rather than PSL(2, C).

One of the invariants under inner automorphism of a subgroup of SL(2, C) is the field generated by the traces of its elements. We shall prove in this paper that this invariant depends only on the commensurability class of the group. By constructing an infinity of groups with distinct trace-fields we then infer that an infinity of commensurability classes exist. I owe to W. Thurston the idea of using this method to prove that there are infinitely many classes.

2. Generic pairs of 2×2 matrices. A peculiar situation usually arises if two projective transformations of the line have a fixed point in common, or, if we think linearly rather than projectively, if two matrices share an eigenvector. In this context, the following theorem collects a number of more or less well-known results, scattered throughout the literature. For any matrix A, $\operatorname{tr} A$ denotes its trace.

2.1 THEOREM. *Let $A, B \in \mathrm{SL}(2, \mathbf{C})$ and let $\operatorname{tr} A = \alpha$, $\operatorname{tr} B = \beta$, $\operatorname{tr} A^{-1}B = \gamma$. Let $Q(x, y, z)$ denote the ternary quadratic form*

$$Q(x, y, z) = \det(xI + yA + zB) = x^2 + y^2 + z^2 + \alpha xy + \beta xz + \gamma yz.$$

Let $\delta = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 4$. Then the following are equivalent:

2.1.1 $\delta = 0$.
2.1.2 *$Q(x, y, z)$ is singular.*
2.1.3 $\operatorname{tr}(ABA^{-1}B^{-1}) = 2$.
2.1.4 *$AB - BA$ is singular.*
2.1.5 *A and B share an eigenvector.*
2.1.6 *The matrices I, A, B, AB are linearly independent.*

Moreover
2.1.7 *The discriminant of Q is $-\frac{1}{4}\delta$.*

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