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A SYMPLECTIC RIGIDITY THEOREM

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§0. Introduction. Any piece of hypersurface in a symplectic manifold can be mapped locally onto any other one by a symplectic diffeomorphism, so there is no local notion of convexity in symplectic geometry. On the other hand, a hypersurface S in a symplectic manifold carries a distinguished line element field $\mathscr{L}_S$, so the global "dynamical" properties of this field could be related to classical geometrical notions such as convexity. For instance, it is known that if S is strictly convex [9] or even just star shaped [5], there is at least 1 closed integral curve for $\mathscr{L}_S$; if S is sufficiently close to an ellipsoid [2] [8], there are at least n.

An example was given in [10] of an embedded 3-sphere in R^4 which could not be made star shaped by any symplectic transformation, leaving open the question of whether star shaped and convex hypersurfaces could be distinguished by purely symplectic properties. In this paper, we present an example of a hypersurface in R^4 which is weakly convex (i.e., with positive semidefinite second fundamental form) but cannot be made strictly convex (i.e., with positive definite second fundamental form) by any symplectic transformation which is C^2 close to the identity. In other words, no symplectic deformation can round out all the flat directions in this hypersurface.

This paper is intended as a small step toward understanding the mostly unknown "flexibility" of symplectic transformations. Most results in this direction, including the present one, apply only to transformations near the identity (see the discussion and references in [11]), and extensions to more general transformations are interesting but very difficult to find. We also note that nothing is known about the possibility of finding symplectic transformations between diffeomorphic *open* subsets of the same volume in R^{2n} for $n \geqslant 2$. (See [4] for $n = 1$ and volume elements.)

§1. Statement of the theorem. Using the coordinates (x_1, x_2, x_3, x_4) on R^4, the symplectic structure may be defined by combining the usual euclidean structure with the linear map J defined by $J(x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$. If $H: \mathsf{R}^4 \to \mathsf{R}$ is a function, its hamiltonian vector field is $J\nabla H$, and if $S \subseteq \mathsf{R}^4$ is an oriented hypersurface with normal field $\mathbf{n}_S$, the distinguished line element field $\mathscr{L}_S$ is generated by $J\mathbf{n}_S$.

Let S_0 be the hypersurface $H_0^{-1}(\frac{1}{2})$, where $H_0(x) = \frac{1}{2}(x_1^2 + x_2^2)$. The field $\mathscr{L}_S$ is generated by the hamiltonian vector field $J\nabla H_0 = x_2(\partial/\partial x_1) - x_1(\partial/\partial x_2)$, all of whose trajectories are closed with length 2π. If S is any hypersurface obtained

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