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ON ALEXANDROV–BERNSTEIN THEOREMS IN HYPERBOLIC SPACE

MANFREDO P. DO CARMO AND H. BLAINE LAWSON, JR.

§0. Introduction. One of the beautiful and intriguing theorems in riemannian geometry is Alexandrov's Theorem which says that a compact hypersurface embedded in R^n with constant mean curvature must be a (round) sphere. This proof extends easily to prove the corresponding result for compact hypersurfaces in hyperbolic n-space H^n. [A]

Another important and beautiful theorem of this type is the Bernstein Theorem (cf. [B], [F], [Al], [dG], [S]) which states that a minimal hypersurface in R^n which admits a 1-to-1 projection onto a hyperplane $\mathsf{R}^{n-1} \subset \mathsf{R}^n$ must also be a hyperplane, when $n \leqq 7$. (This is false for $n \geqq 8$ [BdG].) We shall show (See §3) that an analogous theorem holds in H^n for all n.

Taken together, the Alexandrov and Bernstein Theorems in R^n give a characterization of the (complete) umbilic embedded hypersurfaces among those of constant curvature. In hyperbolic space the umbilic hypersurfaces are somewhat more interesting. There are the *spheres* of constant distance from a point, the *hyperspheres*, of constant distance from a geodesic hypersurface, and the *horospheres*. If we fix the sectional curvature of H^n to be -1, and if $\Sigma \subset \mathsf{H}^n$ is a complete umbilic hypersurface, with normalized mean curvature H, then:

$$\begin{aligned} 1 < H < \infty &\Leftrightarrow \Sigma \text{ is a sphere,} \\ H = 1 &\Leftrightarrow \Sigma \text{ is a horosphere,} \\ 0 \leqq H < 1 &\Leftrightarrow \Sigma \text{ is a hypersphere.} \end{aligned}$$

We now recall that H^n has a natural smooth compactification $\overline{\mathsf{H}^n} = \mathsf{H}^n \cup S^{n-1}(\infty)$ where $S^{n-1}(\infty)$ can be identified with asymptotic classes of geodesic rays in H^n. On his thesis [An] Mike Anderson recently proved that any closed submanifold $B^{p-1} \subset S^{n-1}(\infty)$ is the asymptotic boundary of a p-dimensional minimal variety $\Sigma^p \subset \mathsf{H}^n$. This concept of asymptotic boundary seems crucial in understanding the noncompact umbilic hypersurfaces.

Recall that the sphere $S^{n-1}(\infty)$ carries in a natural way the standard conformal structure. (Isometries of H^n become conformal automorphisms of $S^{n-1}(\infty)$.) The asymptotic boundary of a hypersphere in H^n is a linear hypersphere in $S^{n-1}(\infty)$. All such linear hyperspheres in $S^{n-1}(\infty)$ are conformally equivalent, and we shall refer to them simply as spheres. The points

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