Vol. 50, No. 4 DUKE MATHEMATICAL JOURNAL © December 1983

CHARACTERISTICS AND EXISTENCE OF ISOMETRIC EMBEDDINGS

ROBERT L. BRYANT, PHILLIP A. GRIFFITHS AND DEANE YANG

Let (M^n, ds^2) be an n-dimensional Riemannian manifold. A well-known problem is to prove the existence of a local C^∞ isometric embedding

$$(M^n, ds^2) \hookrightarrow \mathsf{E}^{n(n+1)/2}. \tag{1}$$

By this we mean that there is a smooth isometric embedding of a neighborhood of a given point $x_0 \in M$; to simplify notation, we shall refer to this neighborhood also as M.

When (M, ds^2) is real analytic, the Burstin–Cartan–Janet–Schafly theorem (cf. the references given in [3, 11]) shows that such local isometric embeddings exist.

When $n = 2$ it is also known that local C^∞ isometric embeddings exist in a neighborhood of a point x_0 where the Gaussian curvature $K(x_0) \neq 0$.

When $n \geqslant 2$ it has been proved by R. Greene [7] that local C^∞ isometric embeddings

$$(M^n, ds^2) \rightarrow \mathsf{E}^{(n(n+1)/2)+n}$$

always exist.

In general we may consider the exterior differential system (I, ω) whose integrals give local isometric embeddings

$$(M^n, ds^2) \rightarrow \mathsf{E}^{(n(n+1)/2)+s}. \tag{2}$$

The basic invariant of (I, ω) is its characteristic sheaf $\mathscr{M}$. We may think of $\mathscr{M}$ as a family of vector spaces $\mathscr{M}_{(x,\xi)}$ of varying dimension whose support

$$\operatorname{supp} \mathscr{M} = \{(x, \xi) : \dim \mathscr{M}_{(x,\xi)} > 0\}$$

is the characteristic variety Ξ of (I, ω). However, $\mathscr{M}$ contains much more information, both locally and globally, than Ξ alone. The system (I, ω) is determined when $s = 0$, underdetermined when $s > 0$, and overdetermined when $s < 0$. This is reflected in the properties of $\mathscr{M}$ in a precise way (cf. the appendix to §II(c)).

In particular, let us consider the case $s = 0$. Although the system (I, ω) is only invariant under the group $E(n)$ of Euclidean motions, it turns out that both $\mathscr{M}$

Received December 14, 1982. First author partially supported by NSF Grant MCS 580-03237. Second author partially supported by the Guggenheim Foundation and NSF Grant MCS 81-04249. Third author NSF Predoctoral Fellow.