Vol. 50, No. 3 DUKE MATHEMATICAL JOURNAL © September 1983

VARIETIES OF LOW Δ-GENUS

TONY HOROWITZ

Introduction. Let $C \subset \mathbf{P}^n$ be an irreducible curve not lying in any hyperplane. Then $\deg C \geqslant n$; else a hyperplane through any n points of C must contain C by Bézout. Similarly, if $X \subset \mathbf{P}^n$ is an irreducible variety of dimension r not lying in any hyperplane, then $\deg X \geqslant n - r + 1$; for the section of X by a general $(n - r + 1)$-dimensional linear subspace $\Lambda \subset \mathbf{P}^n$ is an irreducible curve $C \subset \Lambda$ with $\deg C = \deg X$.

Now let V be a smooth, irreducible variety, $r = \dim V$, L a line bundle on V such that $|L| = \mathbf{P}H^0(V, L)$ has no base locus. Let us assume moreover that the morphism $\rho_L : V \to \mathbf{P}^n$ associated to the linear system $|L|$ is birational; here $n = h^0(V, L) - 1$ (we say that L is "birationally very ample"). Let $X = \rho_L(V)$, $d = d(V, L) = c_1(L)^r$; then $d = \deg X$ and $d + r \geqslant h^0(V, L)$ by the previous paragraph. We define

$$\Delta = \Delta(V, L) = d + r - h^0(V, L);$$

Fujita ([F1], [F2]) calls this the Δ-genus of (V, L). (In fact, he shows that $\Delta(V, L) \geqslant 0$ for any ample line bundle L; we will not be using this, however.)

It is well known that if $\Delta(V, L) = 0$, then X is one of the following: (1) $\mathbf{P}^r$ (2) A quadric hypersurface in $\mathbf{P}^{r+1}$ (3) The Veronese surface in $\mathbf{P}^5$ or a cone over it (4) A rational normal scroll (i.e., $(V, L) \cong (\mathbf{P}E, \mathcal{O}_{\mathbf{P}E}(1))$, where E is a vector bundle on $\mathbf{P}^1$ such that E^* has global sections and $\mathcal{O}_{\mathbf{P}E}(1)$ is the birationally very ample tautological line bundle). (See Harris [JH], Fujita [F2].) Fujita ([F3]) and Iskovskih ([I]) have classified (V, L) with $\Delta(V, L) = 1$.

It is worth noting that if $\Delta(V, L) = 1$, then $h^0(V, L) \leqslant 10$. The general fact for regular surfaces V is that if $\Delta = \Delta(V, L) \geqslant 1$, then $h^0(V, L) \leqslant 3\Delta + 6$, with the single exception of the del Pezzo surface in $\mathbf{P}^9$, where $h^0(V, L) = 10 = 3\Delta + 7$. We will review an argument of Harris and Eisenbud for this result in Section 1. Hence, with this exception, a regular surface V with $\Delta(V, L) < \frac{1}{3}h^0(V, L) - 2$ must have $\Delta(V, L) = 0$. Our first objective is the following generalization:

THEOREM A. *Suppose* $r = \dim V = 2$, $(V, L) \not\cong (\mathbf{P}^2, \mathcal{O}(3))$, $\Delta(V, L) < \frac{1}{3}h^0(V, L) - 2$. *Then* $X = \rho_L(V)$ *is projectively ruled, and* $\Delta(V, L) = 2h^1(\mathcal{O}_V) - h^1(V, L)$. *In particular,* $h^1(\mathcal{O}_V) \leqslant \Delta(V, L) \leqslant 2h^1(\mathcal{O}_V)$.

COROLLARY. *Suppose* $r \geqslant 3$, L *very ample, and* $2 \leqslant \Delta(V, L) < \frac{1}{3}(h^0(V, L) - r - 4)$. *Then* $X = \rho_L(V)$ *is projectively ruled* (*i.e. there exists a morphism of* V *to a*

Received January 13, 1983.