Vol. 50, No. 2 DUKE MATHEMATICAL JOURNAL © June 1983

SOME EXAMPLES OF NONSMOOTHABLE VARIETIES WITH NORMAL CROSSINGS

ULF PERSSON AND HENRY PINKHAM

The recent work of Bob Friedman [F] has greatly clarified the study of deformations of reduced varieties X with normal crossings. In particular he exhibits a necessary condition, which he calls d-semistability, for X to be the special fiber of a semistable degeneration with smooth generic fiber over the unit disk, (for short, semistably smoothable) and then constructs for such X an intrinsic limiting mixed Hodge structure which agrees with the usual one if X is semistably smoothable. So in some ways a d-semistable X is as good as a semistably smoothable one.

The question remains, however, whether any d-semistable X is indeed semistably smoothable. The purpose of this note is to show that this is not the case. We will show more: there are d-semistable X with no smooth deformations at all, i.e., even if we drop the condition that the total space of the deformation be smooth, then X has no deformations with smooth generic fiber. The construction is quite simple as the obstruction occurs at the tangent space level.

In the first section we review the definitions we need in our special setting, explain the main construction and then explain the connection with some work of Kodaira [K]; in the second section we make the cohomological computation which shows that the smooth deformations of X are already obstructed at the tangent space level. This computation implies the main result of the first section, which we have preserved mainly for pedagogical and esthetic reasons. We also relate our result to an earlier result of Nakamura [N], giving a sufficient condition for smoothability. Finally in the third section we give some explicit examples.

Both authors would like to thank Bob Friedman for interesting conversations concerning this article.

Notation and conventions. If D is a smooth subvariety of a smooth variety X, then Θ_X is the tangent bundle of X, Ω^1_X is the cotangent bundle, and $N_{D/X}$ the normal bundle of D in X. If $\mathscr{F}$ is a sheaf on X, $\mathscr{F}|_D$ denotes its restriction to D. All our varieties are compact analytic varieties over the complex numbers $\mathbf{C}$. $\mathbf{C}[\epsilon]$ denotes the ring of dual numbers, i.e., $\epsilon^2 = 0$, and $T_\epsilon = \operatorname{Spec} \mathbf{C}[\epsilon]$ will be used as the base for infinitesimal deformations: see for example [B].

Received November 3, 1982. Revision received March 7, 1983. First author supported by the NFR (Research Council of Natural Sciences of Sweden). Second author supported by the NSF under Grant n° MC5-8005802.