Vol. 50, No. 2 DUKE MATHEMATICAL JOURNAL © June 1983

ON EISENSTEIN SERIES

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The groups on which our Eisenstein series will be defined are symplectic and unitary ones, which include as a special case a group G defined by

$$G = \left\{ \alpha \in \mathrm{SL}_{2m}(K) : {}^t\bar{\alpha}\eta\alpha = \eta \right\}, \qquad \eta = \begin{bmatrix} 0 & -1_m \\ 1_m & 0 \end{bmatrix}$$

with an imaginary quadratic field K and a positive integer m. Let H be the domain consisting of all complex square matrices z of size m such that $i({}^t\bar{z} - z)$ is positive definite. Writing a typical element α of G in the form $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with matrices a, b, c, d of size m, we define the action of α on H by $\alpha(z) = (az + b) \cdot (cz + d)^{-1}$ for $z \in H$, and a factor of automorphy j by $j(\alpha, z) = \det(cz + d)$. Let P be the subgroup of G consisting of the elements α for which $c = 0$. Then we consider Eisenstein series of the following two types:

$$E(z, s; k, \Gamma) = \sum_{\alpha \in (P \cap \Gamma) \backslash \Gamma} j(\alpha, z)^{-k} |j(\alpha, z)|^{-s},$$

$$E(z, s; k, \psi, \mathbf{b}) = \sum_{\alpha \in (P \cap \Gamma_{\mathbf{b}}) \backslash \Gamma_{\mathbf{b}}} \psi(\det(d)) j(\alpha, z)^{-k} |j(\alpha, z)|^{-s}.$$

Here $z \in H$, $s \in \mathbf{C}$, $k \in \mathbf{Z}$, Γ is a congruence subgroup of G, $\mathbf{b}$ is an ideal of $\mathbf{Z}$, ψ is a Dirichlet character modulo $\mathbf{b}$, and $\Gamma_{\mathbf{b}}$ is the congruence subgroup of G consisting of all α whose entries are all integral and such that $c \equiv 0 \pmod{\mathbf{b}}$. As will be shown in the text, $\det(d) \in \mathbf{Q}$ for every $\alpha \in G$. We naturally assume that $\psi(-1) = (-1)^k$. These series are convergent for $\mathrm{Re}(s) + k > 2m$. Moreover, by virtue of the result of Langlands [15], they can be continued as meromorphic functions in s to the whole complex plane. To state our problems, let us denote by $E(z, s)$ any one of the series. Then we ask the following questions.

(A) *Is $E(z, s)$ holomorphic in s at $s = 0$?*

(B) *If so, is $E(z, 0)$ holomorphic in z?*

(C) *If this is so too, does $E(z, 0)$ have cyclotomic Fourier coefficients?*

If $m = 1$, group G in this specialized setting coincides with $\mathrm{SL}_2(\mathbf{Q})$. The above questions in this case were answered by Hecke in the well known paper [9]. It should also be noted that if $k > 2m$, E is convergent at $s = 0$, and hence the answers to (A) and (B) are trivially true.

Received October 21, 1982. Supported by NSF Grant MCS-8100744.