Vol. 49, No. 4 DUKE MATHEMATICAL JOURNAL © December 1982

ON THE L-FUNCTIONS OF CANONICAL HECKE CHARACTERS OF IMAGINARY QUADRATIC FIELDS, II

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The purpose of this note is to resolve a question left open in the second-named author's papers [2] and [3]. The main theorem of [3] was concerned with a special class of Hecke characters of imaginary quadratic fields, referred to in that paper as "canonical" Hecke characters. The gist of the theorem was this: The L-function of a canonical Hecke character vanishes at the center of the critical strip if and only if it is forced to do so by its functional equation. However, this statement was proved only up to a finite number of possible exceptions. Furthermore, the method of proof gave no effective procedure for determining whether such exceptions existed, because it depended on Siegel's lower bound for the class number of an imaginary quadratic field. In the present paper we shall circumvent this problem by replacing the use of the Siegel and Burgess estimates with a positivity argument.

To state the result precisely, let K be an imaginary quadratic field with discriminant $-D < -4$ and ring of integers $\mathcal{O}$, and consider a Hecke character χ of K with conductor $\mathfrak{f}$, satisfying

$$\chi(\alpha\mathcal{O}) = \alpha$$

for $\alpha \equiv 1 \bmod \mathfrak{f}$. We assume that χ is equivariant with respect to complex conjugation, that the values of χ on principal ideals lie in K, and that $\mathfrak{f}$ is divisible only by primes of K which divide D. Let $W(\chi)$ be the root number in the functional equation of $L(s,\chi)$.

THEOREM. *$L(1,\chi) = 0$ if and only if $W(\chi) = -1$.*

For facts about Hecke characters which are used in the proof, as well as for applications to elliptic curves with complex multiplication, the reader is referred to [1], [4], [5], and [6]. Here we recall just two well-known facts about Eisenstein series and theta series which we shall need:

(i) For z in the upper half plane, put

$$G(z,s) = \sum_{\substack{m\in\mathbf{Z}\\ n>0}} \left(\frac{-D}{n}\right)(mD\bar{z}+n)|mDz+n|^{-2s} \qquad (\operatorname{Re} s > 3/2),$$

Received April 23, 1982. Research supported in part by NSF Grant MCS 8002559.