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THE POINTWISE FATOU THEOREM AND ITS CONVERSE FOR POSITIVE PLURIHARMONIC FUNCTIONS

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I. Introduction and statement of results.

1.1. Let $u(z)$ be a harmonic function defined in the open unit disc of the complex plane, having the Poisson integral representation

$$u(\mathrm{re}^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\, d\mu(t) \qquad (0 \leqslant r < 1,\ -\pi \leqslant \theta \leqslant \pi), \tag{1}$$

where μ is a finite Borel measure on $[-\pi,\pi]$. A classical theorem of Fatou [1] relates the differentiability properties of μ to the radial behavior of $u(z)$:

THEOREM A (Fatou). *Let u be the Poisson integral of a finite Borel measure μ, as in* (1), *and let $\theta \in [-\pi,\pi]$. If*

(i) $$\lim_{h\to 0} \frac{1}{2h}\int_{\theta-h}^{\theta+h} d\mu(t) = L,$$

where $L \in \mathbf{C}$, then

(ii) $$\lim_{r\to 1} u(\mathrm{re}^{i\theta}) = L.$$

The implication (ii)⇒(i) does not hold in general. (See Loomis [4] for an example.) There is, however, an appropriate tauberian condition on μ for which the converse is valid:

THEOREM B (Loomis). *If μ is a positive finite Borel measure on $[-\pi,\pi]$, and if $0 \leqslant L < \infty$, then statements* (i) *and* (ii) *of Theorem* A *are equivalent.*

Loomis [4] deduced Theorem B from various results in summability theory for Fourier–Stieltjes series. Rudin [5], using a version of Weiner's tauberian theorem, generalized Loomis' result to the setting of positive harmonic functions defined in upper half spaces of euclidean space; he also showed that the condition $L < \infty$ is needed. (When $L = \infty$, (i) still implies (ii), but the converse is false.)

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