Vol. 49, No. 1 DUKE MATHEMATICAL JOURNAL © March 1982

ASYMPTOTIC FORMULAS WITH SHARP REMAINDER ESTIMATES FOR EIGENVALUES OF ELLIPTIC OPERATORS OF SECOND ORDER

HIDEO TAMURA

§1. Introduction. In this paper we will obtain the best possible remainder estimate in the asymptotic formulas for eigenvalues of elliptic operators of second order on $\mathbf{R}_x^n$, $\mathbf{R}_x^n$ being the n-dimensional euclidean space with a generic point $x = (x_1, \ldots, x_n)$. We consider a uniformly elliptic operator with real smooth coefficients of the form

$$A(x, D_x) = \sum_{j,k=1}^{n} D_j a_{jk}(x) D_k + V(x), \qquad D_j = -i\partial/\partial x_j, \tag{1.1}$$

where $a_{jk}(x) = a_{kj}(x)$. Under suitable assumptions on the coefficients, $A(x, D_x)$ admits a unique self-adjoint realization in $L^2(\mathbf{R}_x^n)$. We denote it by A. If $V(x) > 0$ and if $V(x) \to \infty$ as $|x| \to \infty$, then A is positive and it has an infinite sequence of (positive) eigenvalues, $\{\lambda_j\}_{j=1}^{\infty}$, diverging to infinity. Let $N(\lambda)$, $\lambda > 0$, denote the number of eigenvalues less than λ with repetition according to the multiplicities; $N(\lambda) = \sum_{\lambda_j < \lambda} 1$. The aim of the present paper is to derive the asymptotic formula with the best possible remainder estimate for $N(\lambda)$ as $\lambda \to \infty$.

In order to describe the obtained result more precisely, we fix several notations.

Notations. (a) We follow the standard multi-index notations; for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \cdots \alpha_n!, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \quad (\partial_{x_j} = \partial/\partial x_j), \qquad D_x = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} \quad (D_{x_j} = -i\partial_{x_j}).$$

(b) $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$. (c) $\nabla_x \phi = (\partial_{x_1}\phi, \ldots, \partial_{x_n}\phi)$. (d) $\langle x \rangle = (1 + |x|^2)^{1/2}$. (e) For positive functions $f(x,\xi)$ and $g(x,\xi)$, $f \sim g$ means that f/g and g/f are uniformly bounded.

Assumptions. We make the following assumptions on the coefficients of $A(x, D_x)$:

Received July 6, 1981. Revision received October 9, 1981.