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BIRATIONAL MORPHISMS OF SMOOTH THREEFOLDS COLLAPSING THREE SURFACES TO A POINT

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This paper classifies proper birational morphisms of smooth threefolds collapsing three smooth surfaces meeting normally to a point. In addition to three blow-ups and Hironaka's example of the blow-up of two plane curves, one new nonprojective morphism, the wagon wheel, is found, which collapses two smooth surfaces meeting normally to a plane curve with one ordinary singular point. Neither of the surfaces is birationally equivalent to the blow-up of the singular point. Elementary modifications of threefolds are defined and used to describe both of the nonprojective morphisms. A number of formulae and facts related to birational morphisms of threefolds and their factorizations are also established.

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Introduction. The factorization question asks whether proper birational mappings between smooth algebraic spaces over $\mathbf{C}$ may be written as compositions of blow-ups and blow-downs with smooth centers. Hironaka's theorem on elimination of points of indeterminacy (Hironaka [5] and [6]) reduces that question to that of proper birational morphisms. If $f: X \to Y$ is a proper birational morphism, let S_f be the subvariety of Y where f^{-1} is not a morphism and let $E_f = f^{-1}(S_f)$. By Zariski's Main Theorem (Zariski [18] and van der Waerden [16]), E_f is a divisor and S_f has codimension at least two. Thus for curves, such morphisms are isomorphisms. For surfaces, such morphisms have

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