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# ON THE EXISTENCE OF BOUNDED POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN EXTERIOR DOMAINS

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§1. **Introduction.** In this paper we consider a semilinear elliptic boundary value problem of the form

$$\begin{aligned} -\mathfrak{D}u(x) &= f(x,u(x)) && \text{in} \quad \Omega, \\ Bu(\xi) &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\mathfrak{D}$ is a uniformly elliptic second order differential operator with smooth coefficients, $B$ is the usual first order boundary operator, $\Omega$ is the exterior of a bounded domain $D$ in $R^d$ and $f(x,u)$ is a function concave in $u$. In particular we are concerned with the so called unforced case; $f(x,0) \equiv 0$. Then the problem (1.1) obviously has a trivial solution $u \equiv 0$, and our interest rests on the existence or nonexistence of bounded positive solutions, in other words, on the uniqueness or nonuniqueness of bounded nonnegative solutions of the problem (1.1). We shall give criteria for this existence problem for a rather wide class of nonlinear term functions $f(x,u)$ (Theorem 2.2 below). The specific feature of our criteria is that they are given in term of recurrence properties in probability theory and principal eigenvalues in spectral theory.

As in the bounded domain case, our method is fully based on linearization. But we have to divide the problem into two cases; (a) the diffusion $X$ on $R^d$ corresponding to the generator $\mathfrak{D}$ is recurrent and (b) the diffusion $X$ is transient. In the case of (a), the most tedious task is to give a maximum principle for the boundary value problem

$$\begin{aligned} -\mathfrak{D}u(x) - (\lambda + c(x))u(x) &= f(x) && \text{in} \quad \Omega, \\ Bu(\xi) &= 0 && \text{on} \quad \partial\Omega, \end{aligned} \tag{1.2}$$

when the function $\lambda + c(x)$ can take positive values. The main trouble is that the Green operator for the problem (1.2) can neither be expected to be compact nor to be self-adjoint. We can overcome this to some extent by the following two steps: first we construct a positive $\lambda$-harmonic function $\phi_\lambda$ for $\lambda \leqslant \lambda_0 =$ (the principal eigenvalue) (Proposition 4.1), and then we reduce the maximum principle to the one in bounded domains with the help of $\phi_\lambda$ (Lemma 5.3). In the

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