Vol. 48, No. 2 DUKE MATHEMATICAL JOURNAL © June 1981

THE TORELLI SPACES OF PUNCTURED TORI AND SPHERES

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Introduction. The Teichmüller and Torelli Spaces are moduli spaces which parametrize the variations of complex structure on a surface of a given type. In this paper we establish that the Torelli spaces of spheres and tori with punctures are actually certain hyperbolic complex domains which we can describe exactly.

Our method is to define explicitly some holomorphic universal covering spaces where certain complex domains $D_{g,k}(g = 0, 1)$ are covered by the Teichmüller spaces of punctured spheres and tori. It turns out that the domains being covered are the corresponding Torelli spaces.

The case of the spheres was already known to D. Patterson [6]. In the torus case our result comes from an interesting topological theorem characterizing the diffeomorphisms on a punctured torus whose action on homology is the identity.

Since the Teichmüller modular group is the full group of automorphisms on Teichmüller space (except for a few exceptional cases), we are able to exhibit in Section 4 the full holomorphic automorphism groups of the domains $D_{g,k}((g,k), \neq (1,2))$.

Indeed we can express the Torelli modular groups for the punctured tori as groups of integer matrices, and we make a novel application of this information to derive a purely topological theorem in Section 5.

Finally, in Section 6, we gather some information regarding the Kobayashi metric on the hyperbolic domains $D_{g,k}$ by utilizing the Teichmüller metric on Teichmüller space and the perturbation formula for solutions of Beltrami equations.

We need some definitions to start the analysis.

1. The C^∞ approach to moduli theory.

Definition 1.1. Let S be a C^∞ surface. Two smooth Riemannian metrics are called *conformally equivalent* if they are proportional at every point. An equivalence class of metrics is called a *smooth conformal structure* on S.

LEMMA 1.2. *The smooth conformal structures, $M(S)$, on a compact Riemann surface S are in 1-1 correspondence with the smooth Beltrami differentials $\mu = \mu(z)(d\bar{z}/dz)$ (in local coordinates) with $\|\mu\|_\infty < 1$. This gives a natural*

Received November 14, 1980. Revision received February 5, 1981. This is essentially part of the author's 1980 doctoral thesis at Cornell University. The author expresses his gratitude to his thesis advisor, Professor C. J. Earle, for suggesting the problem and for his generous guidance and help.