Vol. 46, No.4 DUKE MATHEMATICAL JOURNAL © December 1979

THE INDEX OF THE HECKE RING, T_k, IN THE RING OF INTEGERS OF $\mathsf{T}_k \otimes \mathsf{Q}$

NAOMI JOCHNOWITZ

Section 1. Introduction. Let k be an even integer and let S_k be the vector space of cusp forms of weight k for the full modular group. Recall the following formula which for any prime p gives the action of the Hecke operator T_p on the q-expansion of an element of S_k.

$$T_p : \sum a_n q^n \mapsto \sum a_{np} q^n + p^{k-1} \sum a_n q^{np}$$

Let $\mathsf{T}_k \subseteq \mathrm{End}_{\mathsf{C}} S_k$ be the commutative ring generated by the Hecke operators. Then it is well known that $\mathsf{T}_k \otimes \mathsf{Q}$ is isomorphic to a direct product of totally real number fields whose dimension over Q equals $\dim_{\mathsf{C}} S_k$. Moreover, in all known cases, $\mathsf{T}_k \otimes \mathsf{Q}$ is actually isomorphic to a single number field.

The ring T_k is isomorphic to an order in the product of number fields $\mathsf{T}_k \otimes \mathsf{Q}$. In other words, T_k is a subring of $\mathsf{T}_k \otimes \mathsf{Q}$ which when considered as a Z module is free of maximal rank. $\mathsf{T}_k \otimes \mathsf{Q}$ has a unique maximal order, isomorphic to the product of rings of integers of its component number fields, and it is interesting to ask how large is the order T_k. More precisely, if $\mathcal{O}_k$ denotes the maximal order in $\mathsf{T}_k \otimes \mathsf{Q}$, one may want to determine the index $[\mathcal{O}_k : \mathsf{T}_k]$.

For those weights k where S_k is one dimensional ($k = 12, 16, 18, 20, 22, 26$) it is easy to see that $\mathsf{T}_k \times \mathsf{Q} \approx \mathsf{Q}$ and $\mathsf{T}_k = \mathsf{Z}$. For $k = 24$, $\mathsf{T}_k \otimes \mathsf{Q} \approx \mathsf{Q}[\sqrt{144169}\,]$, and T_k is the unique suborder of index 24 in the ring of integers of $\mathsf{T}_k \otimes \mathsf{Q}$. ([8] §8)

In this paper we prove the following theorem dealing with the natural question: What happens to the index $[\mathcal{O}_k : \mathsf{T}_k]$ as the weight k approaches infinity?

THEOREM 1.1. *If N is any integer and k is sufficiently large, N divides the index* $[\mathcal{O}_k : \mathsf{T}_k]$.

This is obviously equivalent to

THEOREM 1.2. *Given any prime l and any positive integer N, there exists an integer B such that if $k \geqslant B$, l^N divides the index* $[\mathcal{O}_k : \mathsf{T}_k]$.

We actually prove the stronger result that for any prime l and any positive integer N, if the weight k is sufficiently large, the additive abelian group $\mathcal{O}_k / \mathsf{T}_k$ contains a direct sum of N subgroups of order l.

Received July 2, 1979; Revision received August 22, 1979.