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IRREDUCIBLE CHARACTERS OF SEMISIMPLE LIE GROUPS II. THE KAZHDAN-LUSZTIG CONJECTURES

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1. Introduction. Let G be a connected semisimple Lie group. In [15], a study of the irreducible characters of G was begun, using the ideas developed by Jantzen for Verma modules and extended to group representations in [13]. In particular, the explicit determination of these characters was reduced to the problem of decomposing certain representations $\{U_\alpha(X)\}$ attached to an irreducible representation X; by this was meant the determination of the irreducible composition factors of each $U_\alpha(X)$, and their multiplicities, in terms of the Langlands classification of irreducible representations. It was conjectured ([15], Conjecture 3.15) that each $U_\alpha(X)$ is completely reducible.

Since [15] was written, Kazhdan and Lusztig have given in [9] a conjecture for the characters of irreducible quotients of Verma modules. (By [4] or [8], this is equivalent to finding the irreducible characters of a complex semisimple Lie group G). The present paper generalizes this conjecture to representations of real linear groups, and reduces its proof to the complete reducibility conjecture mentioned above; in fact, the conjectured formulas are equivalent to the complete reducibility conjecture. (The assumption of linearity is invoked only in section 7 below, for minor technical reasons; it can almost certainly be eliminated with a little cleverness.)

The idea of the argument is very simple. Let $\mathfrak{g}$ be the complexified Lie algebra of G, θ a Cartan involution, and $\mathfrak{g} = \mathfrak{l} + \mathfrak{u}$ a θ-stable Levi decomposition of a θ-stable parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{g}$. The idea is to try to compute $H^i(\mathfrak{u}, X)$ as an $\mathfrak{l}$ module whenever X is an irreducible Harish-Chandra module. This is done first when X is an irreducible representation induced from a discrete series on a cuspidal parabolic subgroup of G (Theorem 6.13). Once the answer is known for some X, one can compute $H^i(\mathfrak{u}, U_\alpha(X))$ (Theorem 7.2). If $U_\alpha(X)$ is completely reducible, this information (via [14]) allows one to determine the irreducible constituents of $U_\alpha(X)$ and their multiplicities (Proposition 5.5); and induction can proceed. The (conjectured) character formulas can be computed from the multiplicities in the various $U_\alpha(X)$, as in [15], or can be written in terms of the $H^i(\mathfrak{u}, X)$ by an Euler-Poincaré principle (Theorem 8.1—see also the remarks at the end of section 8).

Because the computations in general are quite complicated, we will first treat the Verma module case, assuming the infinitesimal character to be integral. Most of the ideas are already apparent there.

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