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A NEW PROOF OF CERTAIN FORMULAS FOR p-ADIC L-FUNCTIONS

NEAL KOBLITZ

The purpose of this paper is to give simple proofs using p-adic measures for Leopoldt's formula $L_p(1, \chi)$ [13] and for formulas expressing $L_p'(0, \chi)$ and $L_p(k, \chi)$, $k \geqslant 1$, in terms of the p-adic log Γ-function. The expression for $L_p'(0, \chi)$ was derived by J. Diamond in [2] and by B. Ferrero and R. Greenberg in [5], and the formula for $L_p(k, \chi)$ was proved in [3].

§1. Construction of p-adic L-functions. Let $\chi : (\mathbf{Z}/d\mathbf{Z})^* \to \overline{\mathbf{Q}}^*$ be a primitive Dirichlet character, which we consider as a function $\chi : \mathbf{Z} \to \overline{\mathbf{Q}}^*$ by extending by zero to n not prime to d. Recall ([6]) that if $L(s, \chi) = \sum \chi(n) n^{-s}$ is analytically continued to the complex s-plane, we have

$$L(1-n, \chi) = -\frac{1}{n} B_{n,\chi}, \tag{1.1}$$

where $B_{n,\chi}$ is defined by

$$\sum_{0 \leqslant a < d} \frac{\chi(a) t e^{at}}{e^{dt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}. \tag{1.2}$$

(The usual Bernoulli numbers B_n are defined by (1.2) with $d = 1$, $\chi(0) = 1$.) Thus, $L(1-n, \chi)$ is in the field $\mathbf{Q}(\chi)$ obtained by adjoining the values of χ. It is these algebraic special values of $L(s, \chi)$ which can be studied arithmetically, and used to form the p-adic analog of $L(s, \chi)$ for any prime p.

Let $\mathbf{Z}_p$, $\mathbf{Z}_p^* = \mathbf{Z}_p - p\mathbf{Z}_p$, $\mathbf{Q}_p$, Ω_p denote, respectively, the ring of p-adic integers, the multiplicative group of units in $\mathbf{Z}_p$, the field of fractions of $\mathbf{Z}_p$, and the p-adic completion of the algebraic closure $\overline{\mathbf{Q}}_p$. The functions $\mathrm{ord}_p\, x$ and $|x|_p$ on Ω_p are normalized so that $\mathrm{ord}_p\, p = 1$. Fix an imbedding $\overline{\mathbf{Q}} \subset \Omega_p$; any algebraic number will be considered simultaneously complex and p-adic via this imbedding. In particular, the same letter χ will denote the character on $(\mathbf{Z}/d\mathbf{Z})^*$ considered to take values in $\mathbf{C}$ or Ω_p. For any $a \in (\mathbf{Z}/p\mathbf{Z})^*$, let $\omega(a)$ be the unique $(p-1)$st root of 1 in $\mathbf{Z}_p^*$ which reduces to a modulo p. ω is a primitive Dirichlet character mod p, which we may also consider as a function on $\mathbf{Z}$ or $\mathbf{Z}_p$ (defined to be 0 on $p\mathbf{Z}_p$). Let χ_n denote the primitive character induced by $\chi\overline{\omega}^n$ ($\overline{\omega}$ denotes the complex conjugate character $\overline{\omega} = \omega^{-1}$).

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