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ON THE 2-CONNECTEDNESS OF VERY AMPLE DIVISORS ON A SURFACE

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In a recent paper, which appears in this journal (vol. 46, no. 2, pp. 337–401), A. Sommese has studied—for good reasons, see Corollary IV below—the following

PROBLEM. *Let $\mathcal{L}$ be a very ample line bundle on a (complex) algebraic surface X, and let $\mathcal{K}_X$ be the canonical bundle of X.*

(a) *When is $\mathcal{L} \otimes \mathcal{K}_X$ spanned by global sections?*

(b) *When is $\mathcal{L} \otimes \mathcal{K}_X$ very ample?*

This problem can of course be studied starting from the classification of surfaces, but in this note I would like to show that an easy and satisfactory answer to (a) can be obtained by using some ideas developed by Bombieri in [1]. In fact, I shall prove:

THEOREM I. *Let D be a very ample divisor on a (complex) surface X. Then D is 2-connected if and only if D is not one of the following:*

(i) *a sum of two lines in $\mathbf{P}_2$ (possibly coinciding)*

(ii) *the sum of one section and a number of fibres (some of which may coincide) on a ruled surface.*

Remark. A very ample divisor is always 1-connected.

THEOREM II. *Let $\mathcal{L}$ be a very ample line bundle on a (complex) surface X. Then $\mathcal{L} \otimes \mathcal{K}_X$ is spanned by its sections if and only if $\mathcal{L}$ is not one of the following:*

(i) $\mathcal{O}(1)$ *or* $\mathcal{O}(2)$ *on* $\mathbf{P}_2$;

(ii) *a line bundle on a ruled surface, the restriction of which to a fibre has degree* 1.

Theorem II was conjectured by Sommese (loc.cit.). N. B. In my terminology, a ruled surface is a $\mathbf{P}_1$-bundle over a smooth curve, and not any blown-ups of such surfaces.

The same method also enables me to give a partial answer to (b); for this see Theorem III below.

Since this note rests completely on the concept of an n-connected divisor and on Ramanujam's vanishing theorem, I recall ([1], p. 177):

An effective divisor D is called (numerically) *n-connected* if for every splitting $D = D_1 + D_2$, with D_1 and D_2 effective, the inequality $D_1 D_2 \geqslant n$ holds.

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