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HYPERPLANE SECTIONS OF PROJECTIVE SURFACES I—THE ADJUNCTION MAPPING

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Classically, the adjunction process was introduced by Castelnuovo and Enriques [C + E] to study curves on ruled surfaces. This paper grew out of an attempt to understand the process geometrically and to find proofs I could understand of the classical results of Enriques [E] on surfaces with hyperelliptic hyperplane sections. Precisely, I ask the following questions.

QUESTIONS. *Let L be a very ample line bundle on a smooth connected projective surface, X.*

(A) *When is $K_X \otimes L$ spanned by global sections?*

(B) *Assuming that $K_X \otimes L$ is spanned, what is the structure of the map, $\phi_L : X \to \mathbf{P}_{\mathbf{C}}$, associated to the space, $\Gamma(K_X \otimes L)$, of sections of $K_X \otimes L$?*

(C) *Is the above map, which I call the adjunction mapping, well behaved enough to be used to classify hyperplane sections of projective surfaces?*

The answers turned out to be better than I had hoped.

Before I discuss the answers, let me mention some related maps. In my work [So$_1$, So$_2$] on the period mapping, I constructed maps using $\Gamma(K_X \otimes \mathfrak{L})$ for appropriate line bundles $\mathfrak{L}$; this construction is very natural from the viewpoint of curvature. Iitaka [I$_1$, I$_2$], in analogy with the canonical mapping, defined and studied the meromorphic map associated to $\Gamma((K_X \otimes L)^n)$ for $n \gg 0$ and L the line bundle associated to a general divisor. Sakai [Sa$_1$, Sa$_2$, Sa$_3$] made a thorough study of when $(K_X \otimes L)^n$ is spanned by global sections for $n \geqslant 3$, where L is the line bundle associated to a rather general curve. He also studied the structure of the associated mapping for large n. Griffiths and Harris [G + H] worked with $K_X \otimes L^2$ for residue reasons.

The answer to (A) is:

(1.5) THEOREM. *$\Gamma(K_X \otimes L)$ spans $K_X \otimes L$ if and only if $h^{1,0}(X) \neq g$ where g is the genus of a smooth $C \in |L|$.*

Let me discuss the history of this result. In the original version of this paper I conjectured the above result and proved it under a variety of additional hypotheses, e.g., $h^{1,0}(X) = 0$, or g being a prime, or $C \cdot C > g + 1$ for $C \in |L|$, and others. Then Van de Ven [VdV] by an entirely new method gave the *first* proof of (1.5) and showed it equivalent to a higher form of connectedness for $C \in |L|$. In revising the original version for publication I realized that my original proof with one modification actually proved (1.5) without the additional

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