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COVARIANT REPRESENTATIONS ON THE CALKIN ALGEBRA I

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0. Suppose $\mathcal{H}$ is a Hilbert space and G is a locally compact group with a strongly continuous unitary representation on $\mathcal{H}$. Then G acts by conjugation on the bounded operators $\mathcal{L}(\mathcal{H})$ and on the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K} = \mathcal{K}(\mathcal{H})$ is the ideal of compact operators. Suppose also that X is a G-space, so that G acts on the continuous functions $C(X)$ by $f_g(x) = f(g^{-1}x)$.

Our intention is to study covariant representations $\tau : C(X) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where covariance means that $g \cdot \tau(f) \cdot g^* = \tau(f_g)$. Examples indicate that the problem should be modified to study covariant representations $\tau : C(X) \to \mathcal{L}_G/\mathcal{K}$ where $\mathcal{L}_G = \{T \in \mathcal{L}(\mathcal{H}) \mid \text{the function } g \rightsquigarrow gTg^* \text{ is norm-continuous}\}$. In this paper we lay the foundations for the study of such representations.

We begin by establishing general properties of $\mathcal{L}_G$. It is a C^*-algebra containing the scalars and $\mathcal{K}$. Further, the operators fixed by the action of G have a particularly simple form relative to the decomposition of $\mathcal{H}$ into G-invariant subspaces.

Second, we study lifting questions. Suppose $a \in \mathcal{L}_G/\mathcal{K}$, where a is fixed and has some additional property. Then one may ask if a can be lifted to $A \in \mathcal{L}_G$ which is fixed and has the same property. If a is self-adjoint, it lifts; if a is a projection and G is compact, then a lifts; if a is unitary and G is compact, then under the additional hypothesis that $\mathrm{ind}_G(a) = 0$, a lifts to a fixed unitary.

Third, we study the topology of the G-Fredholm operators and prove analogs of Atkinson's theorem and Kuiper's theorem in the equivariant case.

The fourth type of result deals with the structure of $\mathcal{L}_G$ as a C^*-algebra. The most striking case is when G is the circle $\mathbf{T}$ represented faithfully on $L^2(\mathbf{T}) \otimes l_2$. Then $\mathcal{L}$ is generated as a C^*-algebra by the fixed operators and by one additional operator B, a block bilateral shift. The structure of $\mathcal{L}_G$ is explained for G compact abelian and for G compact. We plan to consider $\mathcal{L}_G$ for G locally compact (abelian) in subsequent work.

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