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AN ANALOGUE OF WEYL'S THEOREM FOR UNBOUNDED DOMAINS, II

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Introduction. In Part I the authors developed an analogue for exterior regions of a classical theorem of H. Weyl on the asymptotic distribution of the eigenvalues for interior problems. If the region is exterior to a strictly convex body, K, in R^n with n odd, then the analytical quantity defined by the *change* from $-\lambda$ to λ of the *argument of the determinant* of the *scattering matrix*, $\Delta_{\lambda,\lambda}$ arg det $\mathscr{S}(\lambda)$, behaves like twice the counting function for the eigenvalues for interior problems. More precisely, the principal result in [6] is

$$\frac{1}{2\pi}\Delta_{-\lambda,\lambda} \arg\det \mathscr{S}(\lambda) \sim C(n)\mathrm{Vol}(K)\lambda^n + O\left(\lambda^{n-\frac{1}{5}+\epsilon}\right)$$

where $C(n) = 2(\text{Volume of } S^{n-1})(2\pi)^{-n}(n)^{-1}$. The authors also conjectured that the above formula is true provided that the exterior region has no trapped ray paths of geometrical optics.[1]

Here we will give further evidence that $\frac{1}{2\pi}\Delta_{-\lambda,\lambda}$ arg det $\mathscr{S}(\lambda)$ behaves like twice the counting function for interior eigenvalues. We will also develop a suggestive (but by no means well understood!) link between these calculations and the type of calculation used for interior asymptotics (see [1] and [3]).

In the first section of this paper, we study asymptotic formulas for $\Delta_{-\lambda,\lambda}$ arg det $\mathscr{S}(\lambda)$, where $\mathscr{S}$ is the scattering matrix associated with the wave equation $u_{tt} = \Delta_g u$. Here

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{g}\right)\frac{\partial}{\partial x_j},$$

$g^{ij} = (g_{ij})^{-1}$, $g = \det g_{ij}$ and $g_{ij}(x) = \delta_{ij}$ for $|x| > \rho$. The operator Δ_g is the Laplace-Beltrami operator for the Riemannian metric defined by

$$ds^2 = g_{ij}dx^i dx^j.$$

In this situation the geometrical quantity corresponding to Vol(K) is the dif-

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[1](Added in proof): This result for the case of star-like bodies and other extensions of Part I are contained in *Asymptotic behavior of the scattering phase for exterior domains* by A. Jensen and T. Kato (U. C. Berkeley preprint, 1978).