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ON THE TORSION IN $K_4(\mathbb{Z})$ AND $K_5(\mathbb{Z})$

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Introduction

For any commutative ring Λ with unit, Quillen [13] has defined the algebraic K-groups $K_q(\Lambda)$, $q \geq 0$. This definition agrees with that given by Bass [1] when $q = 1$ and Milnor [12] when $q = 2$. These groups have been computed when Λ is a finite field (see Quillen [16]) and when $\Lambda = \mathbb{Z}$ and $q \leq 3$. (See Bass [1], Milnor [12], Karoubi [8], and Lee-Szczarba [9], [11].) It is also known that $K_q(\mathbb{Z})$ is finite except when $q = 0$ or $q > 4$ and $q \equiv 1 \bmod 4$ in which case the rank of $K_q(\mathbb{Z})$ is one. (See Borel [3].) The purpose of this paper is to prove the following result.

THEOREM 1. *The groups $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$ contain no p-torsion for primes p greater than five.*

We note that C. Soulé [19] has extended our arguments to show that $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$ contain no 5-torsion.

In the course of proving this result, we are able to determine the cohomology of $SL_4(\mathbb{Z})$ with coefficients in particular domains. Explicitly, we prove the following.

THEOREM 2. *Let Λ be a ring in which the primes 2, 3, and 5 are invertible. Then*

$$H^q(SL_4(Z); \Lambda \simeq \Lambda \quad \text{if} \quad q = 0, 3,$$
$$= 0 \quad \text{otherwise.}$$

The proof of our first theorem suggests the possibility that, for any prime p, $K_n(\mathbb{Z})$ has p torsion only if the group $GL_n(\mathbb{Z})$ has p-torsion. Now $GL_n(\mathbb{Z})$ contains the symmetric group on $n + 1$ letters so it has p-torsion for $p \leq n + 1$. On the other hand, if $g \in GL_n(\mathbb{Z})$ has order p, the roots of the characteristic polynomial of g are p^{th} roots of unity so $n \geq p - 1$. We thus have the following:

CONJECTURE. *The group $K_n(\mathbb{Z})$ has p-torsion, p a prime, only if $p \leq n + 1$.*

We note that Lichtenbaum [21] has conjectured that, for m even,

$$|\zeta(-m)| = \frac{\text{order } K_{2m}(\mathbb{Z})}{\text{order } K_{2m+1}(\mathbb{Z})}$$

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