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DOMINATION OF SEMIGROUPS AND GENERALIZATION OF KATO'S INEQUALITY

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§1 Introduction

Let (Ω, Σ, μ) be a measure space and consider the corresponding real and complex L^2-spaces $\mathscr{K} = L^2_{\mathbb{R}}(\Omega, \Sigma, \mu)$ and $\mathscr{H} = L^2_{\mathbb{C}}(\Omega, \Sigma, \mu)$. $\mathscr{K}$ becomes an ordered Hilbert space (see e.g. Faris [7]) through the proper, closed cone $\mathscr{K}^+$, consisting of all functions in $\mathscr{K}$ which take (μ-essentially) nonnegative values. For any f in $\mathscr{H}$ or $\mathscr{K}$ we set

$$|f|(\omega) = |f(\omega)| \tag{1}$$

and

$$(\operatorname{sign} f)\,(\omega) = \begin{cases} f(\omega)/|f(\omega)| & \text{if } f(\omega) \neq 0 \\ 1 & \text{else} \end{cases} \tag{2}$$

which gives a polar decomposition

$$f = \operatorname{sign} f \cdot |f| \tag{3}$$

A contraction semigroup $\exp - tH$ on $\mathscr{H}$ is said to be positivity preserving, if it maps $\mathscr{K}^+$ into itself. In the selfadjoint case, Simon [15] recently showed that this property holds precisely if H obeys Kato's inequality, i.e. if $f \in \mathscr{D}(H^{1/2})$ implies $|f| \in \mathscr{D}(H^{1/2})$ and for $f \in \mathscr{D}(H)$, $g \in \mathscr{D}(H^{1/2}) \cap \mathscr{K}^+$

$$\operatorname{Re}\langle H^{1/2}g, H^{1/2}|f|\rangle \leq \operatorname{Re}\langle (\operatorname{sign} f)\, g, Hf\rangle \tag{4}$$

In the proof he made use of a certain equivalent quadratic criterion going back to Beurling and Deny [3], [4]. Related results had been obtained by Aronszajn-Smith [1]. Simon also generalized this criterion to the situation where two self-adjoint semigroups $\exp - tH$ on $\mathscr{H}$ and $\exp - tK$ on $\mathscr{K}$ are related to each other by the domination relation.

$$|(\exp - tH)\, f| \leq \exp - tK|f| \qquad f \in \mathscr{H} \tag{5}$$

and showed that this implies $|f| \in \mathscr{D}(K^{1/2})$ if $f \in \mathscr{D}(H)$ as well as

$$\langle K^{1/2}g, K^{1/2}|f|\rangle \leq \operatorname{Re}\langle(\operatorname{sign} f)\, g, Hf\rangle$$
$$f \in \mathscr{D}(H), \qquad g \in \mathscr{D}(K^{1/2})$$

He then conjectured that conversely this infinitesimal condition would imply inequality (5).

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