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LOCALLY BOUNDED TOPOLOGIES ON GLOBAL FIELDS

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Let R be a ring and let $\mathcal{T}$ be a ring topology on R (that is a topology making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R). A subset A of R is *bounded* for $\mathcal{T}$ if for each neighborhood U of zero, there exists a neighborhood V of zero such that $VA \subseteq U$ and $AV \subseteq U$. $\mathcal{T}$ is a *locally bounded topology* on R if there exists a fundamental system of neighborhoods of zero for $\mathcal{T}$ consisting of bounded sets.

An *algebraic number field* is a finite extension of the rational field Q. An *algebraic function field* is a finite extension of the field $F_q(x)$ of rational functions over a finite field F_q of q elements. A *global field* is either an algebraic number field or an algebraic function field. Shanks and Warner [11] identified all locally bounded topologies on the rational field. We will identify all locally bounded topologies on an algebraic function field and all locally bounded topologies on an algebraic number field K save those that are so strong that the integral closure R of the integers Z in K is an unbounded neighborhood of zero (Corollary 5 of Theorem 3). As a consequence of these results, we obtain a theorem by Mahler [7] concerning normable topologies on an algebraic number field (Corollary 1 of Theorem 3).

However, algebraic number fields and algebraic function fields share sufficiently many properties in common that a unified treatment is possible. These properties concern not only a global field K, but also a Dedekind domain R of which K is the quotient field and a finite set $\mathcal{P}_\infty$ of absolute values on K not defined by prime ideals of R. In §1, we will identify those properties; in §2 we will investigate locally bounded topologies on a field K for which a Dedekind subdomain R and a finite set of absolute values satisfying these conditions have been specified.

1. Basic properties of global fields. Here, K is the quotient field of a Dedekind domain R that is not a field, $\mathcal{P}$ is the set of nonzero prime ideals of R, and $\mathcal{P}_\infty$ is a set $\{|\cdot\cdot|_1, \cdots, |\cdot\cdot|_n\}$ of n mutually inequivalent proper absolute values on K such that for each $k \in [1, n]$ and each $p \in \mathcal{P}$, the topology $\mathcal{T}_k$ defined by $|\cdot\cdot|_k$ is distinct from the topology $\mathcal{T}_p$ defined by the valuation v_p arising from the prime ideal p. We denote $\mathcal{P} \cup \mathcal{P}_\infty$ by $\mathcal{P}'$. For each subset S of $\mathcal{P}'$, we define $0(S)$ by

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