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# LECTURES ON SPECTRAL THEORY OF ELLIPTIC OPERATORS

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The material in the second and third lectures stems from some conversations I had with Iz Singer and Michael Atiyah several years ago in which they called my attention to certain formal similarities between the trace formula which Duistermaat and I had obtained for the spectrum of the Laplace operator and their generalized Lefschetz formulas. I am also indebted to Bert Kostant for pointing out to me that "quantization" is in principle independent of polarization. This idea is behind the attempt, in lecture 3, to formulate the Selberg trace formula in terms of an intrinsic polarization.

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## Introduction

Perhaps the best way to approach the subject matter of these lectures is from the perspective of physics. In classical physics the states of a physical system are points on a symplectic manifold (phase space) and the dynamics of the system are described by a Hamiltonian function, $H$ (the energy function). The symplectic vector field $\xi_H$ associated with $H$ describes how the points of the system move in time. In quantum physics, on the other hand, the states of a physical system are described by vectors in a Hilbert space, and the dynamics of the system by a self-adjoint operator, $H_q$. The motion of the system in time is described by the unitary group $\exp \sqrt{-1}\, tH_q$. A central preoccupation of physics ever since the late twenties has been how to reconcile these two pictures. Here I want to concentrate on one particular aspect of this problem. In the classical picture an *equilibrium state* is a point, $p$, in phase space such that the trajectory of $\xi_H$ which passes through it is periodic. In the quantum picture, an equilibrium state is an eigenstate, $V$, of the operator, $H_q$. The classical system oscillates, in the state, $p$, with a period of oscillation equal to the period of the trajectory through $p$, and in the quantum picture it oscillates with a period $2\pi/\lambda$ where $\lambda$ is the eigenvalue of the eigenstate, $V$. How are these two periods

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