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A SMOOTH CURVE IN $\mathbf{R}^4$ BOUNDING A CONTINUUM OF AREA MINIMIZING SURFACES

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1. Introduction.

We give an example of a smooth closed curve B in $\mathbf{R}^4$ bounding a continuum (homeomorphic to $\mathbf{R}$) of (unoriented) surfaces of least area. As is well known, any closed curve in $\mathbf{R}^n$ bounds a surface of least area, and there are examples of curves in $\mathbf{R}^3$ bounding several area minimizing or minimal surfaces. (Nitsche [11, pp. 396–397] gives several references.) Fleming [7] gives an example (pictured by Almgren [3, p. 3]) of a rectifiable curve (necessarily not smooth) bounding uncountably many minimal surfaces. (See also Lévy [9, p. 29], Courant [5, pp. 119–122], Nitsche [11, pp. 396–398].) But our curve B seems to be the first example of a smooth curve in $\mathbf{R}^n$ bounding infinitely many. Our method is to show that an area minimizing surface with boundary B cannot be invariant under a certain circle of isometries of $\mathbf{R}^4$ which leaves B invariant. Although the result can be proved by several area estimates, we employ a more elegant and general approach using regularity theory.

2. Definitions.

In general we use the terminology of Federer's treatise [6].

Identify $\mathbf{R}^2 \cong \mathbf{C}, \mathbf{R}^4 \cong \mathbf{C}^2$.

Let R be a positive number large enough to insure that $R^2 > 2\pi R + 1$. Define

$$f : \mathbf{C} \to \mathbf{C}^2, \qquad f : z \mapsto (z, Rz^4);$$
$$\mathbf{S}^1 = \{z \in C : |z| = 1\}; \qquad B = f(\mathbf{S}^1).$$

For $\alpha \in \mathbf{S}^1$, let

$$H_\alpha = \{(z, w) \in \mathbf{C}^2 : w \neq 0, w/|w| = \alpha\}.$$

Then if $\beta^4 = \alpha$, $B \cap H_\alpha = \{(\beta, \alpha R), (i\beta, \alpha R), (-\beta, \alpha R), (-i\beta, \alpha R)\}$.

Define a topological isomorphism

$$g : \mathbf{S}^1 \to \Gamma \subset \mathbf{U}(2), \qquad g : u \mapsto g_u ,$$
$$g_u(z, w) = (uz, u^4 w).$$

Clearly B is invariant under Γ.

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