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A COMPACTNESS THEOREM FOR FUCHSIAN GROUPS OF THE SECOND KIND

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1. Introduction.

A Fuchsian group Γ is a discrete subgroup of $G = PSL(2, \mathbf{R})$. Γ acts on the upper half plane U, or equivalently on the unit disc U_0, as a group of conformal automorphisms. Γ also acts as a group of isometries of the Poincaré metric. We will use this metric throughout unless otherwise indicated.

The purpose of this paper is to generalize the following Theorem to Fuchsian groups of the second kind.

THEOREM 1. (Bers [3], Mumford [18]). *The set of conjugacy classes* $[\Gamma]$ *in* G *of Fuchsian groups* Γ *such that* (*i*) area $(U/\Gamma) \leq K < \infty$, *and* (*ii*) $|\text{trace}\, \gamma| \geq 2 + \epsilon > 2$ *for every hyperbolic element* γ *of* Γ, *is compact.*

To motivate our generalization, we interpret Theorem 1 geometrically in terms of the Riemann surfaces U/Γ. (i) implies that the surfaces are compact except possibly for punctures and that the number of punctures and ramification points as well as the genus remain bounded. (ii) means that closed geodesic curves (or geodesic slits connecting ramification points of order 2) are not pinched, i.e., do not have arbitrarily small lengths.

We wish to consider analogous families of surfaces, which, in addition to punctures, may have closed conformal discs removed. Hence, (i) becomes a condition on $N(\Gamma)/\Gamma$ where $N(\Gamma)$ is the Nielsen convex region of Γ. Since we will permit ideal boundary curves to "pinch" to punctures, we require (ii) only for closed geodesics (and geodesic slits) not deformable into ideal boundary curves on U/Γ—{ramification points}. Lastly, we must also require (iii) that the ideal boundary curves themselves do not become arbitrarily "long". Theorem 2 states that (i)–(iii) now characterize compact sets.

The above remarks suggest that the topology we use is more geometric than conformal. Indeed, we can have arbitrarily close surfaces which are at infinite Teichmüller distance (see Bers [3]). Furthermore, our topology is weaker than a Chabauty type topology. Because we allow ramification points of order n to become punctures as $n \to \infty$, a surface need not have a neighborhood consisting of surfaces represented by isomorphic Fuchsian groups (see Harvey [8] and Mumford [18]). However, Theorem 2 can be used to characterize neighborhoods of ideal boundary points of the various moduli spaces (see Abikoff [1] and references given there).

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