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LOCAL TIMES FOR REAL AND RANDOM FUNCTIONS

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Summary. The concept of local time, or occupation-time density, is studied for a real function $x(t)$, t real, and characterized by the integrability of a class of functionals of the form $\psi(|x(s) - x(t)|)$ where $\psi(0) = \infty$ and ψ decreases. (A subsidiary result may be of independent interest: $\psi(|x(s) - x(t)|)$ is non-integrable over the unit square for *any* $x(\cdot)$ and any such non-integrable ψ.) We then show that for functions with a local time, any "approximate local modulus" (in particular, any ordinary modulus of continuity) must grow at least linearly (locally), and faster than linearly if the local time is continuous in its time parameter.

The real variable results are then applied to the trajectories X_t of a Gaussian process. We give a number of examples of Gaussian processes whose trajectories are highly irregular and have no local time; in particular, time-changes of Brownian motion by strictly increasing, continuous (non-random) functions. For Gaussian orthogonal increments processes in general, we give a necessary condition for local time which is very close to a known sufficient condition. We also discuss a conjecture for a necessary and sufficient condition for any Gaussian process to have a local time. Finally, we give several illustrations of how such classical but slightly obscure concepts as approximate limits are well-suited to the study of random processes.

§0. Introduction. Let $x(t)$ be a real-valued Borel function of $t \in U = [0, 1]$ which we regard as the path of a particle. The amount of time spent by the particle in a set B in $\mathcal{B}$ (the Borel σ-field of the real line $\mathbf{R}$) during the interval $[0, t]$ is $\mu_t(B) = m(x^{-1}(B) \cap [0, t])$, m = Lebesgue measure. We say that $x(t)$ satisfies condition (LT), or simply "is (LT)", if the particle spends (Lebesgue measure) zero time in any set of measure zero, i.e. if μ_1 is absolutely continuous ($\mu_1 \ll m$). In this case $\mu_t \ll m$ for every $t \in U$ and a version of the Radon-Nikodym derivative $d\mu_t/dm$ is given by a function $\alpha_t(y)$, $y \in \mathbf{R}$, which may be chosen right-continuous, non-decreasing in t, and $\mathcal{U} \otimes \mathcal{B}$-measurable ($\mathcal{U}$ the Borel σ-field of U) as a function of (t, y). This version is called the *local time at* y and may also be construed as a measure $\alpha(y, dt)$ on $\mathcal{U}$. Similarly, we will say that $x(t)$ is (LT) on a set $E \in \mathcal{U}$ if the measure $B \to m(x^{-1}(B) \cap E)$ is absolutely continuous.

Local times have been extensively studied for trajectories of random processes: for Brownian motion and diffusions by Itô and McKean [14], McKean [16], and

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