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PROPER G_a-ACTIONS

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Introduction.

Let X be an algebraic variety over the field k on which the additive algebraic group G_a acts non-trivally. A necessary condition for the existence of a geometric quotient of X by G_a is that all orbits have the same dimension, which, since the action is non-trivial, can be expressed by saying that G_a has no fixed points in X, or that the isotropy groups in G_a of points in X are all finite. Either of these conditions is equivalent to the statement that the morphism $\theta : G_a \times X \to X \times X$ given by $\theta(t, x) = (tx, x)$ has finite fibres, i.e., is a quasi-finite morphism.

This necessary condition for the existence of a quotient is not usually sufficient, as example 1 below shows. In this paper, we study the case where the action is such that the map θ above is actually a finite morphism (so the action is proper [2, p. 10]). It turns out that this is a sufficient condition for the existence of a quotient, although not a necessary one, as example 2 below shows.

We begin by showing that if X is quasi-affine and a separated quotient exists, then the quotient is quasi-affine. This allows an application of a result of Seshadri which guarantees that if the action is proper and X is normal and quasi-affine, then the quotient exists, and is separated. If k has characteristic zero, then we show that in fact the action is locally trivial. Finally, if X is an open subset of a factorial affine variety and if the action on X is locally trivial for the radiciel topology, then the action must be proper. However, if X is not contained in a factorial affine, the action need not be proper, even if it is locally trivial, as example 2 shows. In example 3 we show that a proper action in positive characteristic need not be locally trivial for the radiciel topology.

Notations and Conventions. Throughout k denotes an algebraically closed field of arbitrary characteristic. A prevariety X over k is a reduced irreducible algebraic k-scheme. A variety is a separated prevariety. Varieties and prevarieties will be identified with their sets of closed points (endowed with the Zariski topology). The structure sheaf of X, denoted $\mathcal{O}_X$, is identified with the sheaf of germs of k-valued rational functions on X.

If R is a finitely generated k-algebra we write Specm R for the affine algebraic set whose coordinate ring is $R/\text{nil}\ (R)$ where nil (R) is the nil-radical of R. If X is a variety and f a regular function on X we let X_f denote $X - f^{-1}(0)$. All algebraic groups are affine.

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