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ON THE L^p NORMS OF STOCHASTIC INTEGRALS AND OTHER MARTINGALES

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1. Introduction. Let X_t, $0 \leq t < \infty$, be standard Brownian motion. It has recently been proved that there exist absolute positive constants A_p, $0 < p < \infty$, and a_p, $1 < p < \infty$, such that if T is stopping time for X_t then

$$E\,|X_T|^p \leq A_p E T^{p/2}, \qquad 0 < p < \infty, \tag{1.1}$$

and

$$a_p E T^{p/2} \leq E\,|X_T|^p, \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad ET^{p/2} < \infty. \tag{1.2}$$

For the exponents $p > 1$ these inequalities are due to D. L. Burkholder in [4] and P. W. Millar in [11]. Inequality (1.2) was extended to the exponents $0 < p \leq 1$ independently by Burkholder and R. F. Gundy in [6] and A. A. Novikov in [13]. The paper [5] is a good general source of information about these and related results.

Here a proof of (1.1) and (1.2) is given which yields the best possible values for the constants a_p and A_p. For $p = 2n$, n an integer, they are respectively z_{2n}^{*2n} and z_{2n}^{2n}, where z_{2n}^* and z_{2n} are the smallest and largest positive zeros of the Hermite polynomial of order $2n$. For $p = 4$ this has already been proved by Novikov in [14], and it is well known that the best values for a_2 and A_2 are 1. Bounds for a_p and A_p may be found in [5], [6], [8], [14], and [16]. The constants found here will be shown to be best possible in inequalities related to (1.1) and (1.2) involving stochastic integrals, stopped random walk, and Haar series.

For example, let $\varphi_1, \varphi_2, \cdots$ be the complete orthonormal system of Haar functions on the Lebesgue unit interval. Let $\lambda_1, \lambda_2, \cdots$ be real numbers, such that $\sum_{i=1}^{\infty} \lambda_i \varphi_i$ converges. Let $f = \sum_{i=1}^{\infty} \lambda_i \varphi_i$ and $S(f) = (\sum_{i=1}^{\infty} (\lambda_i \varphi_i)^2)^{1/2}$. Then there are constants d_p and D_p such that

$$\int_0^1 |f|^p\,dx \leq D_p \int_0^1 S(f)^p\,dx, \qquad 0 < p < \infty, \tag{1.3}$$

and

$$d_p \int_0^1 |f|^p\,dx \leq \int_0^1 S(f)^p\,dx, \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad \int_0^1 S(f)^p\,dx < \infty. \tag{1.4}$$

For the exponents $p > 1$ these inequalities are due to R. E. A. C. Paley [12], who proved an equivalent Walsh series form. Marcinkiewicz [9] noted the

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