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THETA FUNCTIONS WITH COMPLEX MULTIPLICATION

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Dedicated to A. Weil for his 70th birthday

Introduction. The theory of complex multiplication of abelian varieties tells how the value of an abelian function at a point of finite order behaves under Frobenius substitutions. An abelian function can naturally be obtained as the quotient of two theta functions. In the present paper we shall show that a theta function with complex multiplication modified by a suitable exponential factor takes values of algebraic nature at the points commensurable with the periods. The images of the values under Frobenius substitutions can also be determined.

To be more explicit, let V be a vector space over $\mathbf{C}$ of dimension n, and L a lattice in V, i.e., a discrete subgroup of V isomorphic to $\mathbf{Z}^{2n}$. Then we consider a theta function f satisfying

$$\text{(I)} \qquad f(u + l) = f(u)\psi(l)e\left(\frac{1}{2i} H\left(l, u + \frac{l}{2}\right)\right) \qquad (u \in V,\ l \in L)$$

with a hermitian form H and a certain map ψ of L into $\mathbf{C}$, where $e(x) = e^{2\pi i x}$ for $x \in \mathbf{C}$. Assume that $\psi(l)$ for each $l \in L$ is a root of unity, and put

$$\text{(II)} \qquad f_*(u) = e\left(\frac{i}{4} H(u, u)\right)f(u).$$

Then it can easily be verified that

$$\text{(III)} \qquad f_*(u + l) = \psi(l)e(\mathrm{Im}\,(H(l, u))/2)f_*(u) \qquad (l \in L).$$

This implies that f_* can be extended to a continuous function on the adelization of $\mathbf{Q}L$. Suppose that V/L has many complex multiplications. Then V/L determines a number field K' over which the values of abelian functions on $\mathbf{Q}L$ are of abelian nature. Now we call f *arithmetic* if $f_*(u)$ is abelian over K' for every $u \in \mathbf{Q}L$. We shall show that the space of theta functions satisfying (I) can be spanned by arithmetic ones. Moreover, we shall define a natural action of the idele group of K' on the arithmetic theta functions. The exact statement of the properties of the action will form our main theorem in §2.

If we regard a theta function as a function in two kinds of variables, one in the complex vector space and the other in the Siegel upper half space $\mathfrak{H}_n$, then $f_*(u)$ for $u \in \mathbf{Q}L$ is the value of a modular form at a point on $\mathfrak{H}_n$ corresponding to V/L. This fact is essential in the proof of the theorem. Indeed, the

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