Vol. 43, No. 3 DUKE MATHEMATICAL JOURNAL© September 1976

THE DILATATION OF AN EXTREMAL QUASI-CONFORMAL MAPPING

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0. Introduction.

We are concerned with the problem of characterizing the complex dilatations of extremal quasi-conformal maps of the unit disk D to itself. Fundamental progress on this problem, is represented in the work of Richard Hamilton, Edgar Reich and Kurt Strebel. R. Hamilton has shown, in [2], that a necessary condition for $\kappa(z)(\|\kappa\|_\infty < 1)$ to be the complex dilatation of an extremal quasi-conformal map of D to itself, is that

$$(0.1) \qquad \sup_{f \in H} \frac{\left| \iint_D f(z)\kappa(z)\, dx\, dy \right|}{\iint_D |f(z)|\, dx\, dy} = \|\kappa\|_\infty .$$

In (0.1), H is the set of functions which are analytic and integrable over the unit disk. We refer to (0.1) as *Hamilton's condition* on κ.

In [6], Reich and Strebel show that (0.1) is also a sufficient condition for κ to be the complex dilatation of an extremal quasi-conformal mapping. Thus, the first mentioned characterization has been accomplished at a useful and non-trivial level. Nevertheless other characterizations would be desirable. Hamilton's condition is somewhat difficult to verify or contradict in many particular cases. Obvious functions κ for which (0.1) hold are of the form

$$(0.2) \qquad \kappa(z) = k \frac{\bar{f}(z)}{|f(z)|}, \qquad f \in H.$$

The extremal quasi-conformal maps associated with dilatations of type (0.2) are called *Teichmüller extremals.* Indeed, whenever there is a special function f^* in H, for which we have

$$(0.3) \qquad \iint_D f^*(z)\kappa(z)\, dx\, dy = \|\kappa\|_\infty \iint_D |f^*|\, dx\, dy$$

then $\kappa(z)$ is necessarily of the form (0.2). There are also dilatations which are known to satisfy Hamilton's condition but which are not of type (0.2). For such examples, we refer the reader to [3], [6], [10]. These examples have not been classified, and further discernment would be interesting.

Received October 20, 1975. Revision received April 10, 1976. This research is supported by a grant from the National Science Foundation, MPS 75-07513 and by grant NRC 2057-04 of the National Research Council of Canada