Vol. 43, No. 3 DUKE MATHEMATICAL JOURNAL© September 1976

MAGIC LABELINGS OF GRAPHS, SYMMETRIC MAGIC SQUARES, SYSTEMS OF PARAMETERS, AND COHEN-MACAULAY RINGS

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1. Introduction.

Let Γ be a finite graph allowing loops and multiple edges, so that Γ is a *pseudograph* in the terminology of [5]. Let $E = E(\Gamma)$ denote the set of edges of Γ and $\mathbf{N}$ the set of non-negative integers. A *magic labeling of* Γ *of index* r is an assignment $L : E \rightarrow \mathbf{N}$ of a non-negative integer $L(e)$ to each edge e of Γ such that for each vertex v of Γ, the sum of the labels of all edges incident to v is r (counting each loop at v once only). We will assume that we have chosen some fixed ordering $e_1, e_2, \cdots, e_q$ of the edges of Γ; and we will identify the magic labeling L with the vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_q) \in \mathbf{N}^q$, where $\alpha_i = L(e_i)$.

Let $H_\Gamma(r)$ denote the number of magic labelings of Γ of index r. It may happen that there are edges e of Γ that are always labeled 0 in any magic labeling. If these edges are removed, we obtain a pseudograph Δ satisfying the two conditions: (i) $H_\Gamma(r) = H_\Delta(r)$ for all $r \in \mathbf{N}$, and (ii) some magic labeling L of Δ satisfies $L(e) > 0$ for every edge e of Δ. We call a pseudograph Δ satisfying (ii) a *positive pseudograph*. By (i) and (ii), in studying the function $H_\Gamma(r)$ it suffices to assume that Γ is positive. A magic labeling L of Γ satisfying $L(e) > 0$ for all edges $e \in E(\Gamma)$ is called a *positive magic labeling*. Any undefined graph theory terminology used in this paper may be found in any textbook on graph theory, e.g., [5].

In [14] the following two theorems were proved.

THEOREM 1.1. [14, Thm. 1.1]. *Let Γ be a finite pseudograph. Then either $H_\Gamma(r) = \delta_{0r}$ (the Kronecker delta), or else there exist polynomials $P_\Gamma(r)$ and $Q_\Gamma(r)$ such that $H_\Gamma(r) = P_\Gamma(r) + (-1)^r Q_\Gamma(r)$ for all $r \in \mathbf{N}$.*

THEOREM 1.2 [14, Prop. 5.2]. *Let Γ be a finite positive pseudograph with at least one edge. Then* $\deg P_\Gamma(r) = q - p + b$, *where q is the number of edges of Γ, p the number of vertices, and b the number of connected components which are bipartite.*

For reasons which will become clear shortly, we define the *dimension* of Γ, denoted $\dim \Gamma$, by $\dim \Gamma = 1 + \deg P_\Gamma(r)$. In [14, p. 630], the problem was raised of obtaining a reasonable upper bound on $\deg Q_\Gamma(r)$. It is trivial that $\deg Q_\Gamma(r) \leq \deg P_\Gamma(r)$, and [14, Cor. 2.10] gives a condition for $Q_\Gamma(r) = 0$. Empirical evidence suggests that if Γ is a "typical" pseudograph, then $\deg Q_\Gamma(r)$

Received November 11, 1975.