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MINIMAL PROJECTIONS IN $\mathcal{L}_1$-SPACES

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1. Introduction. This work was motivated by a problem recently posed by Dr. Carl deBoor, namely, "What are the projections of least norm from the Lebesgue space $\mathcal{L}_1[-1, 1]$ onto its subspace π_1 consisting of all first-degree polynomials?" We have answered this question, and in the course of doing so have discovered how minimal projections in such a setting may be characterized. Armed with such characterization theorems one can set out to determine minimal projections for various subspaces. The work involved may, however, be formidable, and one may wish to resort to a numerical procedure which will be the subject of another paper.

The problem of obtaining a minimal projection from a Banach space X to a subspace Y could be viewed as a problem of best approximation in the space $L(X, Y)$ of all bounded linear operators from X to Y. Namely, one seeks a best approximation of the 0-operator in the closed convex set of projections from X onto Y.

The minimal projection problem in $C(T)$ has also been studied, for example in [6], but characterization and unicity questions remain open. The minimal projections from $C[-1, 1]$ to π_n are unknown for $n > 1$. For $n = 1$, the minimal projection is the Lagrange interpolation operator for nodes ± 1. It is an open question whether an analogue of Theorem 2, below, is valid in $C(T)$.

Most of our results are valid for general measure spaces with only mild restrictions. In some of the theorems, we require the duality between $\mathcal{L}_1(T, \Sigma, \mu)$ and $\mathcal{L}_\infty(T, \Sigma, \mu)$. This is valid, for example, if the measure space (T, Σ, μ) is σ-finite. But it is somewhat more general to assume outright that $\mathcal{L}_1^* = \mathcal{L}_\infty$. Another hypothesis which occurs frequently is that the subspace Y of $\mathcal{L}_1$ into which we are projecting is smooth. This is true if and only if each member of $Y \backslash 0$ is almost everywhere different from 0[3].

LEMMA 1. *If* $0 \leq \varphi_n \in \mathcal{L}_\infty$ *for* $n = 1, 2, \cdots$ *then* $\sup \|\varphi_n\|_\infty = \|\sup \varphi_n\|_\infty$. (*The norm is the essential supremum norm.*)

Proof. For all n and t, $\varphi_n(t) \leq \varphi(t) \equiv \sup \varphi_n(t)$. Hence $\|\varphi_n\| \leq \|\varphi\|$ and $\sup_n \|\varphi_n\| \leq \|\varphi\|$. In order to prove that $\|\varphi\| \leq \sup_n \|\phi_n\| \equiv k$, it suffices to show that $\varphi(t) \leq k$ a.e. (almost everywhere). If $k = \infty$, this is trivial. If $k < \infty$, it suffices to show that $\mu\{t : \varphi(t) > k\} = 0$. Now $\varphi(t) > k$ if and

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