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DUALITY FOR P-GROUPS AND THEIR COHOMOLOGY RINGS

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Our major aim here is to interconnect the representation theory of a p-group with its $\mathbf{Z}/p\mathbf{Z}$ cohomology ring; see [1] for a related interconnection. We begin by developing a theory of duality for p-groups which extends the duality that holds for an abelian group. While doing so we decompose the character table of a p-group into smaller tables, one for each element in the dual group of a central subgroup. We obtain orthogonality relations for each of these smaller tables.

Next for each real representation of a finite group we define an Euler class. If the representation preserves orientation, this is a $\mathbf{Z}$ cohomology class, otherwise it is only a $\mathbf{Z}/2\mathbf{Z}$ class. We show that it behaves naturally under group morphisms and direct sums of representations and evaluate it in the case of an elementary abelian p-group.

Combining the first two sections with a result of D. Quillen we construct, via generators and relations, a sub-quotient ring of the $\mathbf{Z}/p\mathbf{Z}$ cohomology ring of a p-group. We prove that it captures a non-zero fraction of the whole cohomology.

Finally we evaluate this theory in three separate cases.

1. Duality for p-groups.

For a finite abelian group Z the collection of one-dimensional complex representations form a group Z^* under multiplication and Z^* is the dual group of Z. For a group G with a non-trivial central subgroup Z, we show how this concept of duality generalizes.

Since we deal with complex representations r of G, we can replace r by $\chi = \text{trace}(r)$, its character. It is essentially a complex valued function on the conjugacy classes of elements of G since $\chi(h^{-1}gh) = \text{trace}\,[r(h)^{-1}r(g)r(h)] = \text{trace}\,[r(g)] = \chi(g)$.

We decompose the irreducible characters of G into classes, one for each element in the dual group Z^* of a central subgroup Z of G. Next we introduce a "layering" of the conjugacy classes of G which is dual to the decomposition of the irreducible characters. Using this we show how the character table of G resolves into smaller tables, one for each element in Z^*.

Now we have an inductive construction of the character theory of G in terms of Z and G/Z and some extra data. Thus the procedure is only of value for

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