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WEAKLY ALMOST PERIODIC FUNCTIONS AND GENERATORS OF INVARIANT FILTERS

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Let G be a locally compact abelian group with dual Γ. For a measure μ on G let μ_d denote its discrete part and $\hat{\mu}$ its Fourier–Stieltjes transform. In [7] Ingemar Wik and I showed that $\hat{\mu}_d(\Gamma) \subset \hat{\mu}(F)^-$, where the bar denotes closure and F is an appropriate subset of Γ. Of course it would seem that F must be large in some sense, and it was only later that I found that the appropriate hypothesis on F was that it generate a translation invariant filter of subsets of Γ, i.e., that

$$\bigcap_{\gamma \in E} F\gamma^{-1} \neq \varnothing \quad \text{for all finite} \quad E \subset \Gamma; \tag{1}$$

more generally we can even take a fixed dense subsemigroup Γ_0 of Γ, and for $F \subset \Gamma_0$ demand only that

$$\bigcap_{\gamma \in E} F\gamma^{-1} \neq \varnothing \quad \textit{for all finite} \quad E \subset \Gamma_0 , \tag{1'}$$

which in the case of a non-discrete group allows F to be a remarkably thin set.

In fact we shall see that then $\hat{\mu} \mid F$ determines $\hat{\mu}_d$ itself and, more generally, for any (even separately continuous) topological group Γ and an F satisfying a two-sided version of (1), for any weakly almost periodic function f on Γ [1, 2, 5] $f \mid F$ determines the almost periodic component f_a of f.

Recall that for such Γ, Ryll–Nardzewski's fixed point theorem [1, 8] provides a doubly invariant mean M on $W(\Gamma)$, the space of weakly almost periodic functions on Γ, and the existence of such a mean implies (and in fact is equivalent to) the direct sum decomposition [1, p. 30; 5, 5.11]

$$W(\Gamma) = W_0(\Gamma) \oplus AP(\Gamma) \tag{2}$$

with $AP(\Gamma)$ the space of almost periodic functions on Γ and $W_0(\Gamma)$ the subspace[1] of all f in $W(\Gamma)$ with $M(|f|^2) = 0$. (All footnotes are printed at end of paper.) (For Γ again the dual of a locally compact abelian group G, $\hat{\mu} = \hat{\mu}_c + \hat{\mu}_d$ is the decomposition of $\hat{\mu} \in W(\Gamma)$ provided by (2) where μ_c is the continuous component of μ.) It should be noted that when Γ is abelian the Markov–Kakutani theorem suffices to yield our invariant mean M, and the proof of our decomposition (2) simplifies; for the reader interested only in Fourier–Stieltjes transforms, (2) can just as well be ignored.

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