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SCHIFFER'S INTERIOR VARIATION AND QUASICONFORMAL MAPPING

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Introduction. The purpose of this paper is to show that Schiffer's interior variation by attaching a cell to a Riemann surface is the same as a quasiconformal variation. Then we show how one can use this observation together with Teichmüller theory to simplify some of the results of Schiffer and Spencer, [12].

One motivation for this paper comes from the following observation. Let $Q(S)$ be all integrable quadratic differentials on S. Let z be a local parameter and $z(p) = 0$. Then if φ is in $Q(S)$ and is expressed in terms of z, the mapping $f_p(\varphi) = \varphi(z(p))$ is well-defined and, in fact, is a continuous linear functional on $Q(S)$. Therefore f_p is a tangent vector to Teichmüller space $T(S)$. It turns out that the most common Schiffer variation, the one which uses the formula $z^* = z + \epsilon^2 z^{-1}$ to collapse $|z| = \epsilon$ to a line segment is, up to a constant multiple, a variation which is tangent to f_p. This fact as well as the ideas in §5 were pointed out to me by John Fay.

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All the lettered theorems, (Theorem A and B in §2 and Theorem C in §4), are part of the folklore of Teichmüller theory. The numbered theorems are the author's.

§1. The Schiffer variation is a μ-variation. The Schiffer interior variation by attaching a cell is obtained in the following way, [12, pg. 310]. Let S be a Riemann surface and C a simple, closed, analytic curve on S which bounds a cell. Suppose V is a simply connected, connected, open set containing C and that z is an analytic homeomorphism from V into $\mathbf{C}$. By the Riemann mapping theorem and the Schwarz reflection principle there is no loss of generality in assuming that z maps C to the unit circle. Let V_1 be an annular neighborhood containing C for which $C \subset V_1 \subset V$. Let $r(z)$ be an analytic function defined in $z(V_1)$. In most applications $r(z)$ will be meromorphic in $z(V)$. To get the Schiffer variation you let $z^* = z + \epsilon r(z)$. For sufficiently small complex numbers ϵ, z^* is univalent in V_1 and $C_\epsilon = z^{-1} \circ z^*(C)$ is a simple, closed, analytic curve in V_1. Let $S' =$ the exterior of C in S and let $S'' =$ interior of C_ϵ in S. The variation S_ϵ is the disjoint union of S' and S'' with

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