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ON A PROBLEM OF JULIA

H. ALEXANDER

1. Introduction. Let $G(z, w)$ be an entire function of two complex variables whose zero set is non-empty, irreducible, and not an affine complex line of the form $z = z_0$. An exceptional point of G is a complex number b such that $G(b, w)$ has no zeros in w. Julia's problem is to characterize the closed set E of exceptional points. Julia ([5], reprinted in [6]) showed that E can be any infinite sequence coverging to infinity and that E cannot contain a continuum. These results led Julia to ask ([5], section 9) whether E can ever be uncountable. Tsuji [10] extended the second result of Julia by showing that a necessary condition for an exceptional set is that it have zero logarithmic capacity. The object of this note is to prove that this is also a sufficient condition. That is, given a closed subset E of zero logarithmic capacity in the complex plane, we shall produce a G for which E is the exceptional set. For completeness, we shall also give a simple proof of Tsuji's necessary condition.

The existence of G will be obtained indirectly. We may suppose that E has at least two points. Then, by the uniformization theorem [9], the universal covering surface of $\mathbf{C}\backslash E$ is the open unit disc U. Let $f : U \to \mathbf{C}\backslash E$ be a universal covering map. Our principal result is that there exists a holomorphic function g defined on the unit disc such that $z \to (f(z), g(z))$ is a proper mapping of U into $\mathbf{C}^2$. It follows from the Remmert proper mapping theorem [3] (the special case of this theorem which we need can easily be proved directly) that the image is an irreducible analytic subvariety V of pure dimension one in $\mathbf{C}^2$. From the general theory of several complex variables [3], there is an entire function G whose zero set is V. Clearly, E is then the exceptional set of G.

Finally, I want to thank Allen Shields for a helpful conversation on this material.

2. The sufficient condition. As indicated in the introduction, the problem reduces to proving the following result.

THEOREM. *Let E be a closed subset in the complex plane of zero logarithmic capacity and containing more than one point; let $f : U \to \mathbf{C}\backslash E$ be a universal covering map. Then there exists a holomorphic function g on the unit disc such that $(f, g) : U \to \mathbf{C}^2 (z \mapsto (f(z), g(z))$ is a proper map.*

The proof will be preceded by four lemmas. Let γ be a Jordan curve on the Riemann sphere $\hat{\mathbf{C}}$ which passes through the point at infinity and which

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