A GENERALIZATION AND THE CONVERSE OF COHEN'S FACTORIZATION THEOREM

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1. The converse factorization theorem. A bounded subset U of a Banach algebra A is said to be a bounded (simple) left approximate identity for A if, for arbitrary $\epsilon > 0$ and $a \in A$, there is an element u in U such that $||a - ua|| < \epsilon$. If, for every arbitrary finite subset F of A and $\epsilon > 0$, there is an element u in U such that $||a - ua|| < \epsilon$ for all $a \in F$, then U is called a left multiple approximate identity for A. The following lemma is due to Reiter [10], p. 30.

LEMMA 1.1. *If A has a bounded left approximate identity, then A also has a multiple left approximate identity (possibly unbounded).*

LEMMA 1.2 (*see* [2]). *If U is a bounded left approximate identity for A and $W \subset A$ is a bounded or unbounded multiple left approximate identity for A, then U is a bounded multiple left approximate identity for A.*

Proof. For $\epsilon_0 > 0$ and the arbitrary finite subset F of A, let $w \in W$ be chosen so as to satisfy $||a - wa|| < \epsilon_0$, for all $a \in F$. For $\epsilon_0 > 0$ and w, let $u \in U$ be such that $||w - uw|| < \epsilon_0$. Then we obtain $||a - ua|| \leq ||ua - uwa|| + ||uwa - wa|| + ||wa - a|| \leq ||u|| \cdot ||a - wa|| + ||uw - a|| \cdot ||a|| + ||wa - a|| < d\epsilon_0 + \epsilon_0 \sup_{a \in F} ||a|| + \epsilon_0 < \epsilon$, where $\epsilon > 0$ is arbitrary and ϵ_0 is sufficiently small.

LEMMA 1.3. *If U is a bounded (simple) left approximate identity for A, then U is a bounded multiple approximate identity for A.*

Proof. The proof follows immediately from Lemmas 1 and 2.

Definition. A Banach algebra A possesses the Cohen factorization property if there exists a constant number d such that, for arbitrary $z \in A$ and $r > 0$, there are elements $a, x \in A$ which satisfy the following conditions: (i) $z = ax$, (ii) $||x - z|| < r$ and (iii) $||a|| \leq d$.

LEMMA 1.4. *If A possesses the Cohen factorization property, then A has a bounded left approximate identity.*

Proof. Let $\{r_n\}$ be a sequence of positive numbers converging to 0. Let z be an arbitrary element of A. Then there exist elements a_n and x_n in A such that

$$z = a_n x_n , \quad ||x_n - z|| < r \quad \text{and} \quad ||a_n|| \leq d.$$

Hence, we obtain

$$||z - a_n z|| = ||a_n x_n - a_n z|| \leq ||a_n|| \cdot ||x_n - z|| \to 0 \quad \text{as} \quad n \to \infty.$$

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