ON THE GENERATION OF LINEAR EVOLUTION OPERATORS

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I. Introduction. Let Y be a Banach space with norm $|\ |$, let $R = (-\infty, \infty)$, and let $R^+ = [0, \infty)$. Let $\mathfrak{B}[Y]$ be the algebra of continuous linear functions from Y to Y, I being the identity in $\mathfrak{B}[Y]$, and let $\Delta = \{(t, s) : (t, s) \text{ is in } R \times R \text{ and } s \leq t\}$. By a *linear nonexpansive strongly continuous evolution operator*, hereinafter called an evolution operator, we shall mean a function U from Δ to $\mathfrak{B}[Y]$ such each of properties (P1), (P2), and (P3) hold.

(P1): If (t, s) and (s, r) are in Δ then $U(t, s)U(s, r) = U(t, r)$.

(P2): If $((t, s), x)$ is in $\Delta \times Y$ then $U(t, t)x = x$ and $|U(t, s)x| \leq |x|$.

(P3): If x is in Y then the function from Δ to Y described by $(t, s) \rightarrow U(t, s)x$ is continuous.

The *raison d'être* of the present work is the study of evolution operators, particularly as regards the problem of their generation. In §II, we shall show that with each evolution operator U we can associate a semigroup $\mathcal{S}_U$ over the space $\mathcal{C}_{00}$ (to be defined in §II), and hence with each U we can associate a semigroup generator over $\mathcal{C}_{00}$. A characterization will be given of those semigroups $\mathfrak{I}$ over $\mathcal{C}_{00}$ for which there is an evolution operator U such that $\mathfrak{I} = \mathcal{S}_U$. In §II we shall obtain connections between this concept of generator and the more traditional idea (see, for example, T. Kato [3] and Y. Kōmura [5]) of associating evolution operators with linear time-dependent evolution equations such that

$$u'(r) = A(r)u(r). \tag{1}$$

Our particular interest here is to determine conditions under which product limits such as

$$\lim_{n\to\infty} \prod_{k=1}^{n} \exp\left[\frac{t-s}{n} A\left(\frac{n-k+1}{n} s + \frac{k-1}{n} t\right)\right]x \tag{2}$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left[I - \frac{t-s}{n} A\left(\frac{n-k+1}{n} s + \frac{k-1}{n} t\right)\right]^{-1} x \tag{3}$$

exist and describe an evolution operator. Also in §III, we shall study

$$u'(r) = A(r)u(r) + B(r)u(r),$$

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