FINITE COVERING SPACES OF 2-MANIFOLDS WITH BOUNDARY

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§1. Introduction.

Let S be a compact, orientable 2-manifold with non-empty boundary, and let $p : \tilde{S} \to S$ be an n-sheeted covering space of S. Then $\tilde{S}$ is also a compact, orientable surface with non-empty boundary. What relations must exist between the topological type of S and of $\tilde{S}$? If we denote the Euler characteristics of S and $\tilde{S}$ by χ and $\tilde{\chi}$ respectively then we obviously must have the following equation:

$$\tilde{\chi} = n\chi. \tag{1}$$

Also, if we denote the number of boundary components of S and $\tilde{S}$ by k and $\tilde{k}$ respectively, then the following inequalities must hold:

$$k \leq \tilde{k} \leq nk. \tag{2}$$

Our main theorem is that there are *no* other relations between the topological types of S and $\tilde{S}$. The precise statement is as follows:

THEOREM. *Let S and $\tilde{S}$ be compact, orientable surfaces with non-empty boundaries such that conditions* (1) *and* (2) *hold between the Euler characteristics and the numbers of boundary components respectively. Then there exists a continuous map $p : \tilde{S} \to S$ such that the pair $(\tilde{S}, p)$ is an n-sheeted covering space of S.*

A similar theorem applies if S and $\tilde{S}$ are open (non-compact) orientable surfaces having a finite number of ends, all of which are planar (see chapter I, section 13 of [1] for a discussion of the set of ends of a non-compact surface). Then one must interpret k and $\tilde{k}$ as the number of ends of S and $\tilde{S}$ respectively.

Suppose we define two topological spaces X and Y to be *topologically commensurable* if there exist finite sheeted covering spaces

$$p : \tilde{X} \to X, \quad q : \tilde{Y} \to Y$$

such that $\tilde{X}$ and $\tilde{Y}$ are homeomorphic. Then it is an easy corollary of our main theorem that *any two compact, orientable surfaces having non-empty boundaries and negative Euler characteristic are topologically commensurable.* A similar statement applies to open, orientable surfaces having a finite number of ends, all of which are planar.

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