THE FREE PART OF A TORUS ACTION AND RELATED NUMERICAL EQUALITIES

GLEN E. BREDON

1. Introduction. Let X be a locally compact space. Let G be the toral group $G = T^k$ or the elementary abelian group $G = Z_p^{\,k}$. All cohomology groups will have coefficients in the rationals or in Z_p in these respective cases and are the sheaf theoretic cohomology groups with compact supports. We shall concentrate on the toral case $G = T^k$, and only briefly comment on the modifications needed for the other cases. We assume throughout that $\dim H_c^*(X) < \infty$.

Let $G = T^k$ act on X and consider the open subspace $X_f \subset X$ of points with finite isotropy groups (trivial isotropy when $G = Z_p^{\,k}$). We are interested in calculating the rational cohomology

$$H_c^*(X_f^*)$$

of the orbit space $X_f^* = X_f/G$ of X_f. It is assumed that the cohomology of the fixed set X^T of each subtorus $T \subset G$ is known. We shall show how to calculate this in case X is totally nonhomologous to zero in $X^G = X \times_G E_G$. This is equivalent to the statement that

$$\dim H_c^*(X) = \dim H_c^*(X^G)$$

and will be assumed throughout. We also assume throughout that the number of isotropy groups is finite.

In the toral case, the assumption that X is totally nonhomologous to zero in X_G is equivalent to the degeneration of the spectral sequence of $X_G \to B_G$ and holds, for example, whenever X has zero cohomology in odd dimensions.

Our main theorem yields a collection of k (= rank G) numerical formulas involving the Betti numbers of the fixed sets of subtori of G. The first formula is just the well known Euler characteristic formula $\chi(X) = \chi(X^G)$. The second is a generalization of the Borel formula

$$\dim X - \dim X^G = \sum_{\text{corank } T=1} (\dim X^T - \dim X^G);$$

which was proved by Borel [4] when X is a homology sphere and dim is formal dimension. A version of this for Poincaré duality spaces has been proved by Allday and Skjelbred [2, 10]. Our version also applies to Poincaré duality spaces but gives a somewhat different formula. It is more or less known in the sense that it is implicit in the literature. The third formula is a generalization of Golber's formula [9] for actions on a product of odd dimensional spheres. The general case was proved by Chang and Skjelbred [8]. The rest of the formulas are increasingly more complicated and are new.

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