ON MACKEY SPACES

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The vector spaces we use here are defined over the field K of the real or complex numbers. When we use the word "space" we mean "separated locally convex space". If $\langle E, F\rangle$ is a dual pair we denote by $\sigma(E, F)$, $\beta(E, F)$ and $\mu(E, F)$ the weak, the strong and the Mackey topologies on E, respectively; $\beta^*(E, F)$ is the topology on E of the uniform convergence on every bounded set of E for the $\beta(F, E)$ topology. If a space E has the topology $\mathfrak{J}$ and A is a set of F we shall write sometimes $A[\mathfrak{J}]$ for the set A provided with the topology induced by $\mathfrak{J}$. If B is a bounded closed absolutely convex set in a space E we denote, as usual, by E_B the linear hull of B with the norm associated to B. In a space E we say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence for the Mackey convergence if there is a bounded closed absolutely convex set M of E such that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in E_M. We say that E is locally complete if every Cauchy sequence in E for the Mackey convergence is convergent. It is easy to show that if $\mathfrak{B}$ is the family of all bounded closed absolutely convex sets of E, E is locally complete if and only if E_B is a Banach space for every $B \in \mathfrak{B}$. We denote, as usual, by E' and E^* the topological dual and the algebraical dual of E, respectively. We say that E is a Mackey space if the topology of E coincides with $\mu(E, E')$. If A is a subset of E, we denote by A^* the closure of A in $(E')^*$. $\mathfrak{D}'(\Omega)$ is the space of all distributions of L. Schwartz, with the strong topology, Ω being an open subset of R^n. We denote by ω the cardinal number of the set of the natural numbers.

In ([3], p. 249) it is shown that if E is a sequentially complete space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in E which converges weakly to the origin, then the closed absolutely convex hull of $\{x_n\}_{n=1}^{\infty}$ is weakly compact. The proof given in ([3], p. 249) can be used in part to obtain the following result: a) *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a space E, which converges weakly to x. If A is the closed absolutely convex hull of $\{x_n\}_{n=1}^{\infty}$, then A is weakly compact if and only if E_A is a Banach space.*

THEOREM 1. *Let E be a Mackey space such that $E'[\sigma(E', E)]$ is locally complete. If $E'[\sigma(E', E)]$ is not sequentially complete, there is an one-codimensional subspace F of E, so that F is not a Mackey space.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a non-convergent Cauchy sequence in $E'[\sigma(E', E)]$. Let u be the limit of $\{u_n\}_{n=1}^{\infty}$ in $E^*[\sigma(E^*, E)]$. Let G be the linear hull of $E' \cup \{u\}$. If A is a bounded closed absolutely convex set in $G[\sigma(G, E)]$,

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