CLASSIFYING SPACES FOR SECTIONING MULTIPLES OF A SYMPLECTIC LINE BUNDLE

CHARLES F. PELTIER

The identification of R^{4n} with Q^n (n-dimensional quaternionic space) permits the definition of a fixed-point-free action of $S^3 = S_p(1)$ on $V_{4n,k}$, the Stiefel manifold of k frames in $4n$-space. The orbit space $QV_{4n,k}$ has the following property: If X is a finite complex and a symplectic line bundle over X, then $n\xi$ (Whitney sum) has k linearly independent real sections iff there exists a map $f: X \to QV_{4n,k}$ such that $f^*\gamma_0 = \xi$ where γ_0 represents the symplectic line bundle over $QV_{4n,k}$ associated to the S^3 fibering $V_{4n,k} \to QV_{4n,k}$.

1. Preliminaries. We first establish notation to be used throughout the paper, and then state some useful preliminaries.

We will use $RE(x_i, y_i \mid i \in I)$ to denote the exterior algebra over a ring R with generators x_i, y_i in degree i, and will use $V(x_1, \cdots, x_m)$ to denote the commutative associative algebra over Z_2 with generators $x_1, \cdots, x_m$ such that the monomials $x_1^{\epsilon_1} \cdots x_m^{\epsilon_m}$, $\epsilon_i = 0, 1$, form an additive basis. $C_{a,b}$ denotes the binomial coefficient $\binom{a}{b}$. Let ρ_G denote the universal coefficient map $H^*(\ ; Z) \to H^*(\ ; G)$, shortened to ρ_p for $G = Z_p$. Let $J_{2n,k}$ denote the set of all integers j such that (greatest integer in $((4n - k)/2) < j < 2n$. Let l denote the least element of $J_{2n,k}$ and $t = 2n - l$. We write $H^*(QP^\infty) = Z[\xi]$, $H^*(CP^\infty) = Z[\beta]$, $K^*(QP^\infty) = Z[[z]]$, $K^*(CP^\infty) = Z[[y]]$.

We will use the description of $H^*(V_{4n,k})$ given in [9]. (1.1) $H^*(V_{4n,k}; Z_2) = V(x_{4n-k}, \cdots, x_{4n-1})$ where the subscript denotes degree, and $H^*(V_{4n,k}; Z)$ is generated by the free classes $\bar{y}_{4n-k}$, $\bar{y}_{4n-1}$, y_{4j-1} for $j \in J_{2n,k}$, with $\bar{y}_{4n-k}$ omitted for odd k, and the 2-torsion classes $u_{s,t}$, P^{2j} for $s, t, j \in J_{2n,k}$ and $s < t$. Mod 2 reduction is given by $\rho_2(\bar{y}_i) = x_i$, $\rho_2(y_{4j-1}) = x_{2j}x_{2j-1} + x_{4j-1}$, $\rho_2(P^{2j}) = x_{2j}$, and $\rho_2(u_{s,t}) = x_{2s}x_{2t-1} + x_{2s-1}x_{2t}$.

Now consider the fibration $V_{4n,k} \to E_{4n,k} (= S^\infty \times_{S^3} V_{4n,k}) \to QP^\infty$ induced from the fibration $V_{4n,k} \to BSO(4n - k) \to BSO(4n)$ by the composition $r \circ c \circ n_\gamma$ classifying the n-fold sum of the Hopf bundle γ over QP^∞ regarded as a real vector bundle. By construction $E_{4n,k}$ is the classifying space for finding k real sections of the n-fold sum of a symplectic line bundle over a finite-dimensional complex. The method of [11] gives: Proposition (1.2). The spaces $QV_{4n,k}$ and $S^\infty \times_{S^3} V_{4n,k}$ have the same homotopy type and the natural projection $i: V_{4n,k} \to QV_{4n,k}$ can be identified with the inclusion.

This identification will be implicitly used throughout the paper.

Received February 23, 1974. The material in this paper is contained in a Ph.D. thesis directed by A. Duane Randall at the University of Notre Dame.