A PRIORI BOUNDS FOR POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN TWO VARIABLES

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Introduction. Let $f(x, u)$ be continuous for $0 \leq x \leq 1$ and u real, and satisfy $u^{-1}f(x, u) \to +\infty$ as $u \to +\infty$. Then there is a constant K such that if $-u''(x) = f(x, u)$, $u(0) = u(1) = 0$, and $u > 0$ on $(0, 1)$, then $u(x) \leq K$ on $(0, 1)$. This is a special case of a phenomenon which prevails for a wide class of oscillatory solutions of superlinear Sturm-Liouville problems (cf. [1], [2]). If one considers a corresponding problem in more variables, taking L to be an elliptic operator, say $-\Delta$, in some bounded domain Ω, then superlinear growth of $f(x, u)$ in the variable u will not, in general, guarantee the existence of a priori bounds for positive solutions of the Dirichlet problem for

$$-\Delta u = f(x, u) . \tag{i}$$

If Ω is the unit ball in $\mathbf{R}^3$ and $\lambda > 0$ is appropriately chosen, then there is an infinite sequence of positive, radially symmetric solutions of $-\Delta u = \lambda e^u$ with unbounded L^∞ norms (cf. [3], [4]).

Having a priori bounds, one can derive information about the existence and location of solutions of nonlinear elliptic eigenvalue problems. For the case of one variable this was done for certain classes of nonlinearities in [5]. We leave such applications of bounds in higher dimensions to a separate paper. However, the question remains: for which nonlinearities in which dimensions do a priori bounds exist. The nonexistence of positive solutions of $-\Delta u = u^\beta$ in the unit ball of $\mathbf{R}^n$, established by Pohozaev [6] when $\beta > (n + 2)/(n - 2)$, and the results of Joseph & Lundgren [4] suggest that a priori bounds will not exist in such cases.

Here we examine a particular type of problem: that of power growth of f in the case of two space dimensions. The main results concern the Dirichlet problem for (i) in a bounded, simply-connected domain Ω, having a C^2 boundary $\partial\Omega$. Roughly, we assume that for some $\beta > 1$, $u^{-\beta}f(x, u)$ is bounded above and below by positive constants for large u. In Theorem 72 we obtain L^∞ a priori bounds for positive solutions of (i) provided $1 < \beta < 3$. We have no reason to believe that a priori bounds do not exist for $\beta \geq 3$. In fact, if f does not depend upon x and if Ω is the unit disc in $\mathbf{R}^2$, then in Theorem 39 we show that for any $\beta > 1$ there is an a priori L^∞ bound for a positive, radially symmetric solution of (i).

We also consider the case in which $-\Delta$ is replaced by a general elliptic expres-

Received December 20, 1973. Research supported by NSF Grant GP 32833 and the Italian Government C. N. R.